SAINT IGNATIUS' COLLEGE

Trial Higher School Certificate

2002

MATHEMATICS

EXTENSION 2

9:00am – 12:05 pm
Thursday 29th August 2002

General Instructions

- Reading time : 5 minutes
- Working time: 3 hours
- Write using blue or black pen
- ***Write your name on each answer booklet***
- Board approved calculators may be used
- A table of standard integrals is provided

- Total Marks (120)
- Attempt Questions 1 – 8
- All questions are of equal value

Students are reminded that this is a trial examination only and cannot in any way guarantee the content or the format of the 2002 Mathematics Extension 2 Higher School Certificate examination

Question 1 (15 marks) Use a SEPARATE writing booklet. **Marks**

a) The complex number u is given by $\left(-1 + i\sqrt{3}\right)$.

i) Show that $u^2 = 2\overline{u}$. 2

ii) Evaluate $|u|$ and $\arg u$. 2

iii) Show that u is a root of the equation $u^3 - 8 = 0$. 1

b) If $z = x + iy$ sketch, on separate axes, the locus of z satisfying

i) $\operatorname{Re}(z) = |z|$. 2

ii) Both $\operatorname{Im}(z) \geq 2$ and $|z - 1| \leq 3$. 3

c) Given that both c and d are real numbers, find their values such that 2

$$\frac{c}{1+i} - \frac{d}{1+2i} = 1.$$

d) The points P, Q, R and S on an Argand diagram represent the complex numbers a, b, c and d respectively. 3
If $a + c = b + d$ and $a - c = i(b - d)$, find what type of quadrilateral $PQRS$ is.

Question 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

a) Sketch the following, showing all essential features.

(i) $y = \ln x^2$ 2

(ii) $\sin(x + y) = 1$ 2

(iii) $y = e^x - e^{-x}$ 2

b) (i) Draw (without using the Calculus) a neat sketch of the curve 2

$$y = x^3 - c^2x, \quad \text{where } c \text{ is a positive constant.}$$

Mark clearly any intercepts.

(ii) Use your graph in part (i) to draw neat sketches, on separate number planes, of:

(α) $y = \dfrac{1}{x^3 - c^2x}$ 2

(β) $y = \left|\dfrac{1}{x^3 - c^2x}\right|$ 2

(γ) $y^2 = \dfrac{1}{x^3 - c^2x}$ 3

Marks

Question 3 (15 marks) Use a SEPARATE writing booklet.

a) Find the following indefinite integrals.

(i) $\int 2^{2x}\,dx$ 2

(ii) $\int x e^{x}\,dx$ 2

(iii) $\int \frac{2x}{(x+1)(x+3)}\,dx$ 3

b) By using the substitution $u = t - 4$ evaluate $\int_4^{4\cdot 5} \frac{dt}{(t-3)(5-t)}$. 3

c) (i) If $u_n = \int_0^{\frac{\pi}{2}} x^n . \sin x\ dx \quad , n \geq 2$, 3

prove that $u_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)\,u_{n-2}$

(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} x^2 \sin x\ dx$. 2

Question 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

a) For the hyperbola $\frac{x^2}{20} - \frac{y^2}{5} = 1$, find

(i) the co-ordinates of the two foci, 2

(ii) the equations of the asymptotes 2

b) Explain why $\frac{x^2}{h - 19} + \frac{y^2}{3 - h} = 1$ cannot represent the equation of an ellipse. 2

c) Tangents to the ellipse with the equation $x^2 + 4y^2 = 4$ at the points $A\ (2\cos\theta,\ \sin\theta)$ and $B\ (2\cos\alpha,\ \sin\alpha)$ are at right angles to each other. Show that: $4\tan\theta . \tan\alpha = -1$. 3

d) A and B are variable points on the rectangular hyperbola $xy = c^2$

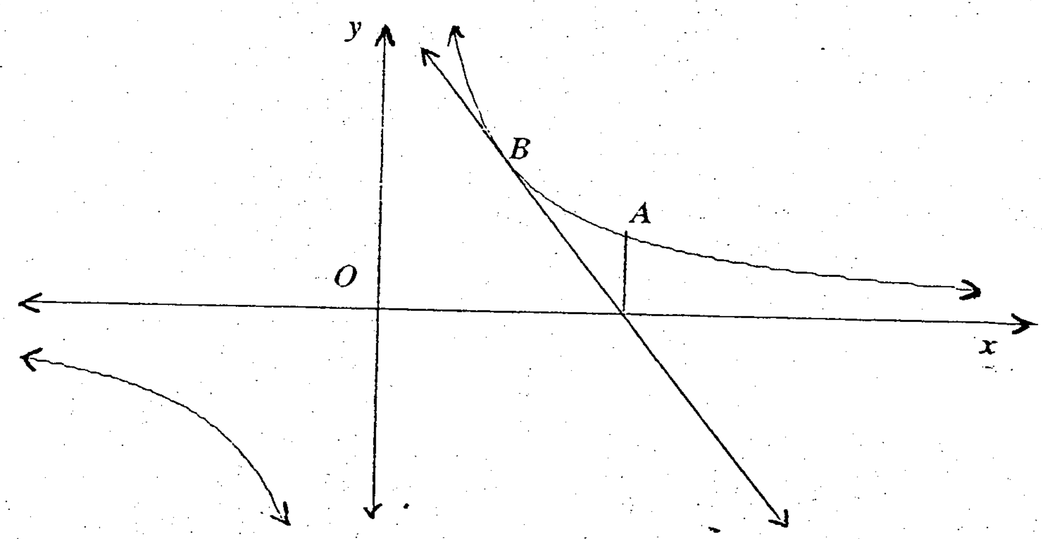

Diagram not to scale.

(i) The tangent at B passes through the foot of the ordinate of A. If A and B have parameters t_1 and t_2, show that $t_1 = 2t_2$ 4

(ii) Hence prove that the locus of the midpoint of AB is a rectangular hyperbola. 2

Marks

Question 5 (15 marks) Use a SEPARATE writing booklet.

a) Prove that both 1 and -1 are zeroes of multiplicity 2 of the polynomial

$$P(x) = x^6 - 3x^2 + 2.$$

Hence express $P(x)$ as a product of irreducible factors over the field of

(i) real numbers **4**

(ii) complex numbers . **1**

b) (i) Assuming the result $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and using the substitution $x = \cos\theta$ solve the equation $8x^3 - 6x + 1 = 0$. **3**

(ii) Hence prove that :

(α) $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{8\pi}{9} = 0$ **2**

(β) $\sec\frac{2\pi}{9} + \sec\frac{4\pi}{9} + \sec\frac{8\pi}{9} = 6$. **2**

c) If α and $-$α are both roots of $x^3 + mx^2 + nx + h = 0$, show that $mn - h = 0$. **3**

Question 6 (15 marks) Use a SEPARATE writing booklet. **Marks**

a) The base of a solid is a right-angled triangle on the horizontal x–y plane; bounded by the lines $y = 0$, $x = 4$ and $y = x$. Vertical cross-sections of the solid, parallel to the y –axis, are semicircles with their diameter on the base of the solid as shown in the diagram below.
Find the volume of the solid. 5

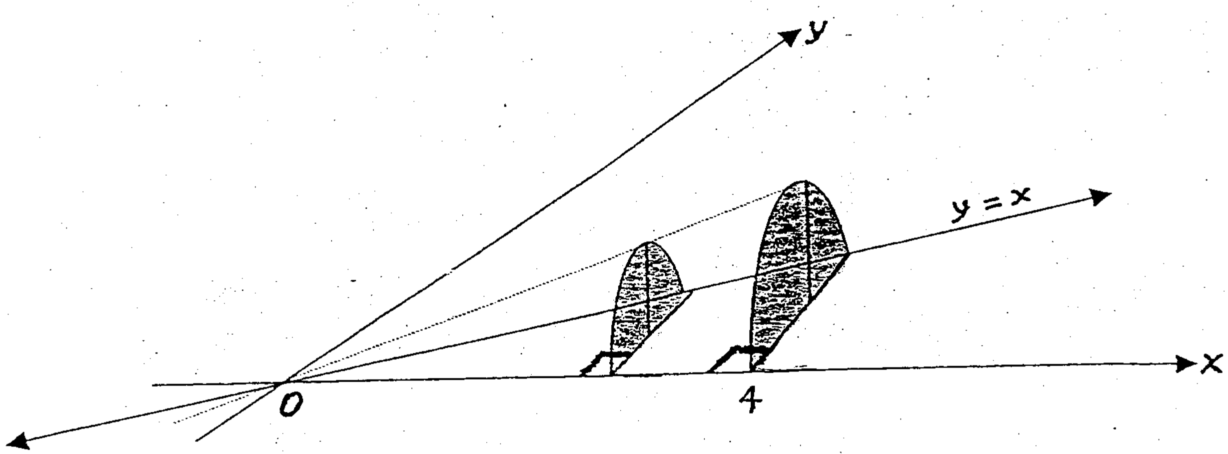

Diagram not to scale

(b) The area bounded by the line $y = 4 - 2x$, the x–axis and the y–axis, is rotated about the line $x = 4$.
By using the method of cylindrical shells find the volume formed. 5

c) Given that for a particular value of x that $\sin^{-1}x$, $\cos^{-1}x$ and $\sin^{-1}(1 - x)$ are acute: 5

(i) Show that: $\sin(\sin^{-1}x - \cos^{-1}x) = 2x^2 - 1$.

(ii) Solve the equation: $\sin^{-1}x - \cos^{-1}x = \sin^{-1}(1 - x)$.

Marks

Question 7 (15 marks) Use a SEPARATE writing booklet.

a) A particle is attached to one end of a light string. The other end is fixed. The particle moves in a horizontal circle (below the fixed point) with a speed of $2\ m\ sec^{-1}$ and the string makes an angle of size $\tan^{-1}\left(\frac{5}{12}\right)$ with the vertical. **4**

Show that the length of the string is approximately 2·5 metres.

Take g as $10\ m\ sec^{-2}$.

b) A particle of unit mass moves in a straight line with variable acceleration $\left(\frac{16}{v} - v\right)\ m\ sec^{-2}$, where $v\ \ m\ sec^{-1}$ is the velocity at time t and $v > 0$, and x is the displacement.

If when $t = 0$, $x = 0$ and $v = 2m\ sec^{-1}$,

(i) Find an expression for the velocity of the particle at time $t\ sec$. **4**

(ii) Find the limiting velocity of the particle. **2**

(iii) Find the displacement of the particle when $v = 3m\ sec^{-1}$. **5**

Question 8 (15 marks) Use a SEPARATE writing booklet. Marks

a)

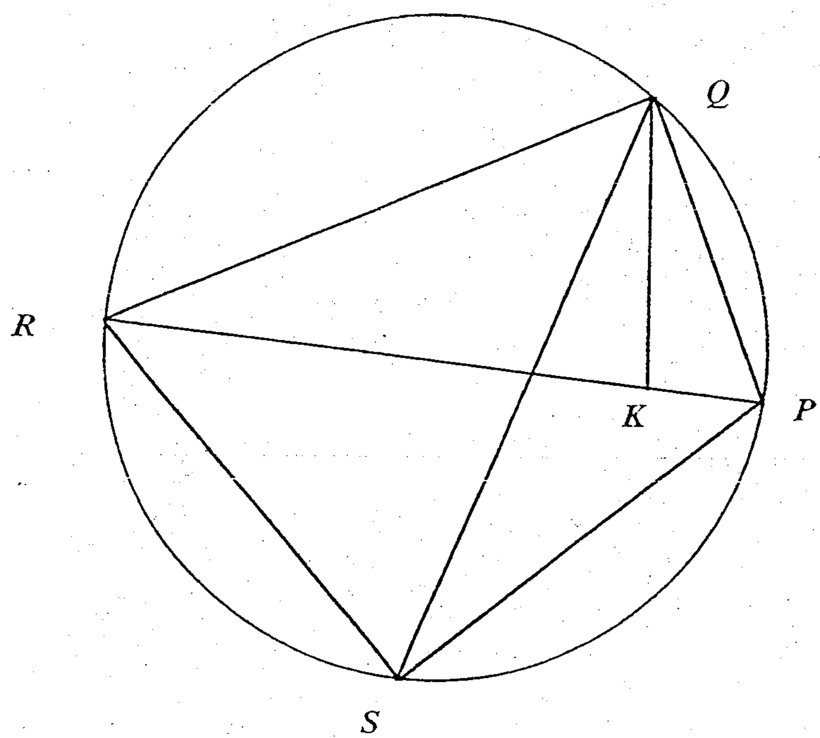

Diagram not to scale.

The above figure is a cyclic quadrilateral. K is the point on RP such that angle PQK is equal to angle SQR.
Let angle $SQR = x^0$ and

(i) Show that triangle PQS is similiar to triangle KQR and that the triangle PQK is similiar to triangle SQR. 6

(ii) Hence show that $PR \cdot SQ = PQ \cdot SR + PS \cdot QR$. 2

b) Prove by Mathematical Induction that: 5

$$\sum_{r=1}^{n} \sin((2r-1)\theta) = \frac{\sin^2 n\theta}{\sin\theta}, \text{ where } n \text{ is a positive integer.}$$

c) For the following statement answer **true** or **false** giving a reason for your answer. 2

$$\text{For } n = 1, 2, 3, \ldots \qquad \int_0^1 \frac{dx}{1+x^n} \le \int_0^1 \frac{dx}{1+x^{n+1}}.$$