ST IGNATIUS COLLEGE RIVERVIEW

ASSESSMENT TASK NUMBER 4

TRIAL HSC EXAMINATION

YEAR 12

2006

EXTENSION 2

Time allowed: 3 hours (+ 5 minutes reading time)

Instructions to Candidates

- Attempt **all** questions
- Show all necessary working.
- Marks may be deducted for missing or poorly arranged work.
- Board approved calculators and templates may be used.
- Each question must be returned in a ***separate*** writing booklet marked Q1, Q2 etc
- **Each booklet must have your name**.

Question 1 [15 marks] **Start a new answer booklet.**

(a) Find $\int \frac{1}{\sqrt{2x-x^2}}\,dx.$ [2]

(b) (i) Resolve $\frac{1}{(x-2)(x+1)}$ into partial fractions. [2]

(ii) Hence, find $\int \frac{1}{(x-2)(x+1)}\,dx.$ [2]

(c) Evaluate $\int_0^m x\sqrt{m-x}\,\,dx$; $x \geq 0,\ m \geq x.$ [3]

(d) Evaluate $\int_0^{\frac{\pi}{3}} \frac{\tan x}{1+\cos x}\,dx,$ using the substitution $t = \tan\frac{x}{2}$. [3]

(e) Evaluate $\int_0^1 \sin^{-1} y\ dy.$ [3]

Question 2 [15 marks] Start a new answer booklet.

(a) Consider the complex number $z = 1 + \sqrt{3}i$.

(i) Find $|z|$ [1]

(ii) Find $\arg\left(\bar{z}\right)$ [1]

(iii) Express $\dfrac{1}{z}$ in the form $a + bi$, where a and b are real. [1]

(b) (i) Find the three cube roots of $-i$ in modulus argument form. [3]

(ii) Represent these three cube roots of $-i$ on an Argand diagram. [1]

(c) Consider the locus in the complex plane defined by $\text{Re}\left(z^2\right) = 4$ where $z = x + iy$.

(i) Find the Cartesian equation of this locus. [1]

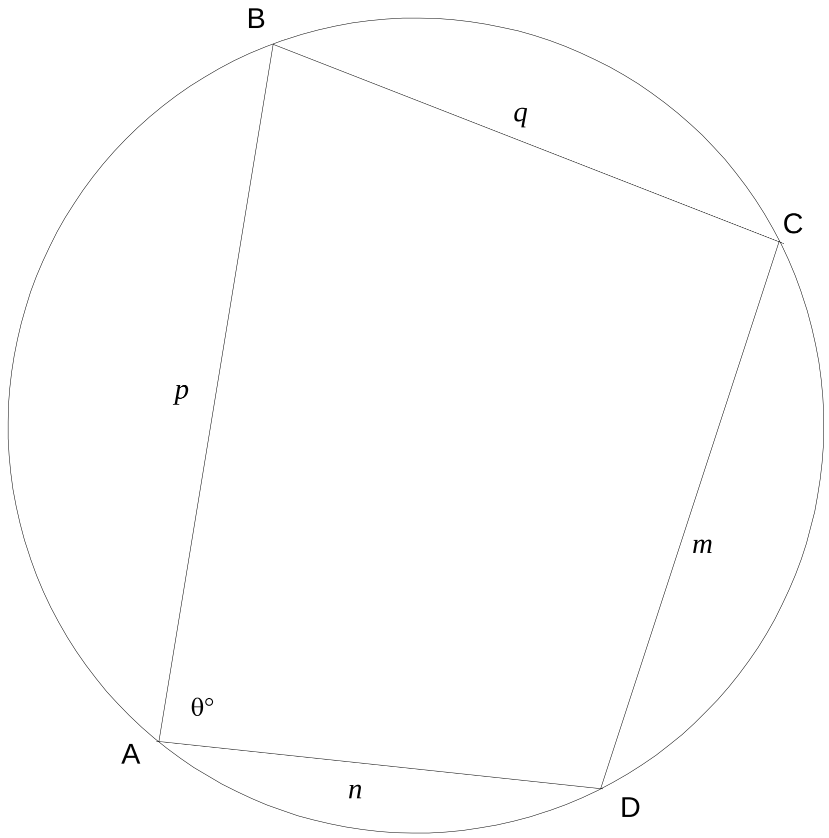
(ii) Describe this locus. [1]

(d) If $\arg\left(z - 3 - 2i\right) = \dfrac{3\pi}{4}$ where $z = x + iy$, sketch the locus of z on an Argand diagram. [2]

Question 2 is continued on the next page

Question 2 CONTINUED

(e) ABCD is a cyclic quadrilateral with sides p, q, m, n as shown in the diagram below.



If the angle DAB is θ degrees:

[2]

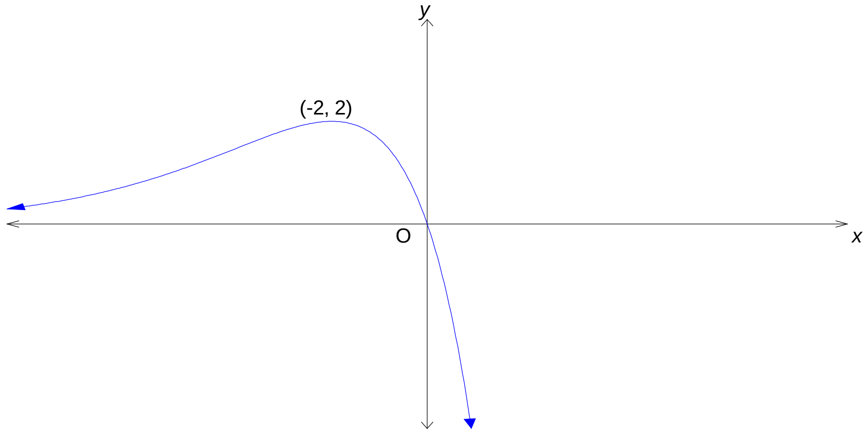
(i) Write down two expressions for the length of diagonal BD.

(ii) Show that [2]

$$\cos\theta = \frac{p^2 - q^2 + n^2 - m^2}{2(pn + qm)}$$

Question 3 [15 marks] Start a new answer booklet.

(a) The curve shown in the diagram is the equation $y=f(x)$. There is a maximum turning point at (-2,2) and the curve crosses the x axis at (0,0).



Sketch the following curves on separate diagrams, showing all of the essential features.

(i) $y=f(x)-2$ [1]

(ii) $y=f(x-2)$ [1]

(iii) $y=f(2x)$ [1]

(iv) $y=f(-x)$ [1]

(v) $y=|f(x)|$ [1]

(vi) $y=\frac{1}{f(x)}$ [1]

(b) Show that the curves $x^2-y^2=c^2$ and $xy=c^2$, where 'c' is a constant, intersect at right angles. [4]

(c) (i) Factorise the expression $(1+x+x^2+x^3)$ over

(α) the real field [1]

(β) the complex field [1]

(ii) Hence, show that the equation $\frac{x^4}{4}+\frac{x^3}{3}+\frac{x^2}{2}+x+c=0$, where '$c$' is a constant, has no real roots if $c>\frac{7}{12}$ [3]

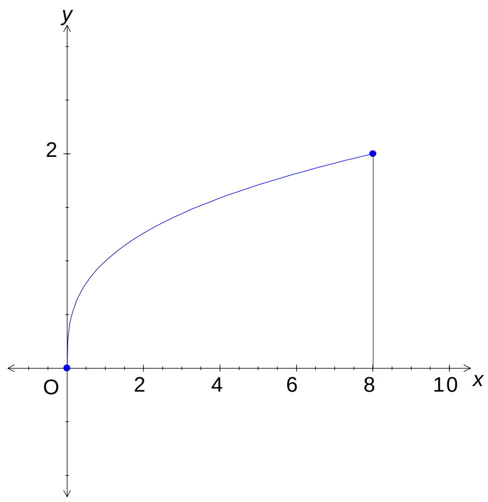
Question 4 [15 marks] Start a new answer booklet.

(a) (i) If $I_m = \int_0^1 x^m e^x \, dx$, where m is a positive integer, show that [3]

$$I_{m+1} = e - (m+1) I_m$$

(ii) Hence, evaluate: $\int_0^1 y^3 e^y \, dy$. [2]

(b) The sketch below shows the region enclosed by the curve $y = x^{\frac{1}{3}}$, the x axis and the ordinate $x = 8$. [4]



Find the volume generated when this region is rotated about the line $x = 8$, using the method of cylindrical shells.

(c) (i) Sketch on the number plane, the circle $(x-2)^2 + y^2 = 9$, stating the co-ordinates of all points of intersection with the axes. [2]

(ii) On this sketch, shade the region bounded by the circle and the y axis where $x \geq 0$. [1]

(iii) The shaded region forms the base of a solid with every cross-section perpendicular to the x-axis forming a square, one side of which lies in the base. Find the volume of the solid. [3]

Question 5 [15 marks] Start a new answer booklet.

(a) Solve the inequality: [3]

$$\frac{x+3}{x-2} > \frac{x+1}{x-3}$$

(b) Divide the polynomial $\left(x^3+5i\,x^2-7i\,x-3\right)$ by $\left(x-2i\right)$ using long division. [3]

(c) Show that $\left(2+\sqrt{3}i\right)$ is a zero of the polynomial $Q\left(x\right)=x^3-x^2-5x+21$. Hence, reduce $Q\left(x\right)$ to factors over the complex field. [3]

(d) The polynomial $P\left(x\right)=x^4+x^2+6x+4$ has a rational zero of multiplicity 2. Find this zero and show that there are no other real zeros. [3]

(e) If $\alpha,\ \beta,\gamma$ are the roots of the equation $x^3+lx+m=0$, where $m\neq 0$, obtain as functions of *l* and *m*, in their simplest form, the coefficients of the cubic equation whose roots are $\frac{1}{\alpha^2},\frac{1}{\beta^2},\frac{1}{\gamma^2}$. [3]

Question 6 [15 marks] Start a new answer booklet.

(a) An engine of mass mkg is travelling with a velocity of v metres / second around a curved banked track of radius R metres, inclined at α degrees to the horizontal plane.

(i) If there is no lateral thrust between the wheels and the rails and N is the normal reaction, prove that: [3]

$\tan\alpha = \frac{v^2}{Rg}$, where g is the acceleration due to gravity.

(ii) If the vintage steam engine "3801" negotiates the track at a speed of 72 kilometres per hour, without exerting lateral thrust on the rails, and the track has a radius of 200 metres and width 1.5 metres, calculate the distance to the nearest centimetre that the outer rail is raised above the inner rail. Take $g = 10$ metres per second squared [2]

(b) (i) A particle of mass m falls from rest, from a point O, in medium whose resistance is mkv, where k is a positive constant and v is the velocity after time t. Prove that the speed at time t is given by [3]

$$\frac{g}{k}\left(1-e^{-kt}\right)$$

(ii) An identical particle is projected upwards from O with initial velocity u in the same medium. If this particle is released simultaneously with the first, prove that the speed of the first particle when the second is momentarily at rest is given by: $\frac{wu}{w+u}$, where w is the terminal velocity of the first particle. [7]

Question 7 [15 marks] Start a new answer booklet.

(a) In the expansion of $(a+b)^{2m+1}$

(i) Find T_{m+1} and T_{m+2} (Note: T means term) [2]

(ii) Hence prove, that if $a+b=1$, then [2]

$$T_{m+1}+T_{m+2}=\binom{2m+1}{m}a^{m}\,b^{m}$$

(b) (i) P $(a\sec\alpha,\, b\tan\alpha)$ and Q $(a\sec\beta,\, b\tan\beta)$ are two points on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where $x>0$.

(γ) Show that the gradient of PQ is : $\dfrac{b\cos\left(\frac{\alpha-\beta}{2}\right)}{a\sin\left(\frac{\alpha+\beta}{2}\right)}$ [3]

(θ) Show that he equation of PQ is : [3]

$$\frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right)-\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha+\beta}{2}\right)$$

Note: The following results may be used for parts (γ) and (θ) above :

$$\sin(A-B)=2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\cos A-\cos B=-2\sin\left(\frac{A-B}{2}\right)\sin\left(\frac{A+B}{2}\right)$$

(ii) If the chord PQ subtends a right angle at the vertex A (a,0) show that $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=1-e^2$. [5]

Question 8 [15 marks] Start a new answer booklet.

(a) By using the substitution $y = \frac{\pi}{2} - x$, show that [7]

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan x}\,dx = \int_0^{\frac{\pi}{2}} \frac{1}{1+\cot x}\,dx$$

Hence find the value of I .

(b) Let $z = x + iy$ be a complex number (x and y are real) satisfying

$$z\bar{z} + (1-2i)z + (1+2i)\bar{z} \le 3$$

(i) Sketch the locus of z on an Argand diagram. [4]

(ii) Find the maximum and minimum values of $(x+y)$. [4]

End of the Paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \; n \neq -1; \; x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$