ST IGNATIUS COLLEGE RIVERVIEW

ASSESSMENT TASK 4

TRIAL HSC EXAMINATION

YEAR 12

2008

EXTENSION 2

Time allowed: 3 hours (+ 5 minutes reading time)

Instructions to Candidates

- Attempt all questions.
- There are eight questions. All questions are of equal value.
- All necessary working should be shown. Full marks may not be awarded if work is careless or badly arranged.
- The questions are not necessarily arranged in order of difficulty. Candidates are advised to read the whole paper carefully at the start of the examination.
- Approved calculators may be used. A table of standard integrals is provided.

- **Each question is to be started in a new booklet. Your number should be written clearly on the cover of each booklet.**

Question 1 [15 Marks] Start a new answer booklet.

(a) Find the following integrals :

(i) $\int \cos^{-1} x \, dx$ [2]

(ii) $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^2 x \, dx$ [2]

(b) (i) Express $\dfrac{25}{(x+2)(2x-1)^2}$ in partial fractions [3]

(ii) Hence show that $\displaystyle\int_1^2 \frac{25}{(x+2)(2x-1)^2} = \frac{10}{3} - 2\ln\frac{3}{2}$ [2]

(c) (i) Using the substitution $x = a - t$, or otherwise,

prove that $\displaystyle\int_0^a f(x)dx = \int_0^a f(a-x)dx$ [2]

(ii) Hence, or otherwise, show that $\displaystyle\int_0^{\frac{\pi}{2}} x(\tfrac{\pi}{2} - x)\sin^2 x \, dx = \frac{\pi^3}{96}$ [4]

Question 2 **[15 Marks]** **Start a new answer booklet.**

(a) If $z = \dfrac{2-i}{1+i}$, where $z = x + iy$, find $\overline{z}$ in the form $(a+bi)$ [2]

(b) Find the square root of $(21+20i)$ in the form $(a+bi)$ [3]

(c) (i) Sketch the locus of $|z+1+i| \le 1$, where $z = x+iy$ [2]

(ii) Find the maximum and minimum values of $|z|$ in part (i) [2]

(d) (i) The complex number $z = x + iy$ is represented by the point P. [3]

If $\dfrac{z-1}{z-2i}$ is purely imaginary, show that the locus of P is a circle, excluding two points.

(ii) State the centre and the radius of this circle. [1]

(iii) Give the co-ordinates of the two excluded points and the reason for their exclusion. [2]

Question 3 [15 Marks] Start a new answer booklet.

(a) Sketch graphs (on separate number planes) of the following relations, without the use of calculus.

Each graph should be labelled clearly.

(i) $y=(x-1)(x+1)$ [1]

(ii) $y=|x-1|(x+1)$ [2]

(iii) $y=\dfrac{1}{(x-1)(x+1)}$ [2]

(iv) $y=\sqrt{(x-1)(x+1)}$ [2]

(v) $y=e^{(x-1)(x+1)}$ [2]

(vi) $y=\log_e(x-1)(x+1)$ [2]

(b) (i) Sketch on the same number plane $y=|x|-2$ and $y=4+3x-x^2$ [1]

(ii) Hence, or otherwise, solve $\dfrac{|x|-2}{4+3x-x^2}>0$ [3]

Question 4 [15 Marks] Start a new answer booklet.

(a) Write down the co-ordinates of the vertices and the foci for the hyperbola $xy = 2$ [3]

(b) P is a point $(a\sec\theta, b\tan\theta)$ which lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, with centre 0.

The tangent P meets the asymptote $y = \frac{b}{a}x$ at Q and the other asymptote at R.

The normal at P meets OQ at K

(i) Represent the above data with a suitable diagram [1]

(ii) Derive the equation of the tangent at P [2]

(iii) Prove that the co-ordinates of Q are [2]

$(a[\sec\theta + \tan\theta], b[\sec\theta + \tan\theta])$

(iv) If the co-ordinates of R are $(a[\sec\theta - \tan\theta], b[\tan\theta - \sec\theta])$ [1]
Prove that P is the midpoint of QR

(v) (α) If P is equidistant from Q, R and O, prove that the hyperbola is rectangular [2]

(β) Hence, prove that $Q\hat{K}P = P\hat{O}R$ [4]

Question 5 [15 Marks] Start a new answer booklet.

(a) Consider the polynomial $P(x) = x^4 - 2x^3 + 2x - 1$

(i) Show that $P(x) = 0$ has a multiple zero and state its value and multiplicity. [3]

(ii) Hence, fully factorise $P(x)$ [2]

(b) Consider the polynomial $f(x) = ax^3 + bx^2 + cx + d$ where a, b, c and d are real. [5]

Given that two of the roots of $f(x) = 0$ are $(1-2i)$ and -2, and that

$f(-1) = -8$, find a, b, c, and d.

(c) If α, β and γ are the roots of the equation $x^3 + 2x^2 - 3x + 4 = 0$ find [5]

A cubic equation whose roots are $\alpha\beta$, $\beta\gamma$ and $\gamma\alpha$.

Question 6 [15 Marks] Start a new answer booklet.

(a) A solid has its base the ellipse $\frac{x^2}{36}+\frac{y^2}{16}=1$ [4]

If each section perpendicular to the major axis is an equilateral triangle,

show that the volume of the solid is $128\sqrt{3}$ cubic units.

(b) The region bounded by the curve $y=\log_e x$, the straight line $x=e$ and the x-axis is rotated about the straight line $x=e$. By taking slices parallel to the x-axis, find the exact volume generated. [5]

(c) Find the exact volume generated, by rotating the area bound by the curves [6]

$y=(x-1)^2$ and $y=x+1$, about the y-axis using the method of

cylindrical shells.

Question 7 **[15 Marks]** **Start a new answer booklet.**

(a)

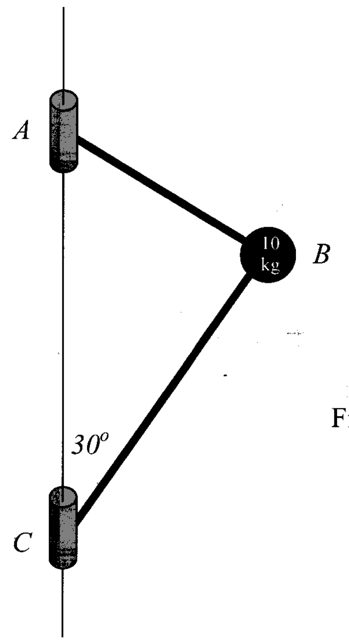

Figure not to scale

The above diagram shows a mass of 10 kilograms at B connected by light rods (at right angles) to sleeves A and C which revolve freely about the vertical axis AC but do not move vertically. The angle between the vertical axis AC and the light rod BC is 30^o. The acceleration due to gravity is g metres per second squared.

(i) Given AC is 2 metres, show that the radius of the circular path of rotation of B is $\frac{\sqrt{3}}{2}$ metres. [1]

(ii) Find the tensions in the rods AB and BC when the mass makes 90 revolutions per minute about the vertical axis. [5]

Question 7 Continued

(b) A particle P of mass m kg projected vertically upward with an initial velocity u metres per second is subjected to forces which create a constant vertical downward acceleration of magnitude g metres per second squared and an acceleration directed against the motion of magnitude kv when the speed is v metres per second squared. K is a constant

(i) Show, with the aid of a diagram, that the acceleration function Is given by $\ddot{x} = -g - kv$ [2]

(ii) Prove that the maximum height reached by the particle after time T is given by $T = \frac{1}{k}\log_e\left|\frac{g+ku}{g}\right|$ [3]

(ii) Prove that the maximum height is $\frac{1}{k}(u - gT)$ [4]

Question 8 [15 Marks] Start a new answer booklet.

(a) (i) Prove by Mathematical Induction that if n is a positive integer, then $2^{(n+4)} > (n+4)^2$ [4]

(ii) By choosing a suitable substitution, or otherwise, show that if a is a positive integer, then $2^{3(a+2)} > 9(a+2)^2$ [2]

(b) (i) Write down the formula for $\tan(A+B)$ in terms of $\tan A$ and $\tan B$ [1]

(ii) Prove that $\tan(2\tan^{-1} x) = 2\tan(\tan^{-1} x + \tan^{-1} x^3)$ [3]

(c) Consider the curve C in the x-y plane defined by $\sqrt{|x|} + \sqrt{y} = 1$

(i) Write down the domain for C [1]

(ii) For $x > 0$, show that $\frac{dy}{dx} < 0$ [2]

(iii) Sketch a graph of C, paying close attention to the gradient of the curve at $x = 0$ [2]

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \; n \neq -1; \; n \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \; x < 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} ax\frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \; x > 0$