St. Scholastica's College

MATHEMATICS EXTENSION 2
April 2003

Time allowed: 3 hours plus 5 minutes reading time

Directions to candidates:
Attempt all questions
Begin all questions on a new page
Show all necessary working
Marks may not be awarded for careless or badly arranged work

NAME:

Topics	Question	Descriptors: O - Outstanding C - Competent D - Developing L - Limited
Integrals (3U) work	1	
Complex Numbers	2	
Graphing Techniques	3	
Conics	4	
Induction, Trig Equation, Calculus (3U)	5	
Conic, Polynomials	6	
Permutations, Combinations Binomial Th'm, Inequality (3U)	7	
Inequalities, Circles, Calculus (3U)	8	

Question 1 (15 marks)

(a) Find $\int_0^2 \sqrt{4-x^2}\,dx$ either from a geometrical diagram or making the substitution $x = 2\sin\vartheta$ 2

(b) Find $\int \frac{\cos^3 x}{\sin^2 x}\,dx$ making the substitution $u = \sin x$ 3

(c) Find $\int \frac{dx}{x(\ln x)^6}$ making a suitable substitution 2

(d) (i) Show that $\frac{3}{t+2} + \frac{2}{t-3} = \frac{5t-5}{(t+2)(t-3)}$ 1

(ii) Hence find $\int_4^5 \frac{(t-1)}{(t+2)(t-3)}\,dt$ correct to 2 decimal places 3

(e) Find the equations of one tangent to the curve $x^2 + xy + y^2 = 7$ at the point on the curve where the gradient is $-\frac{4}{5}$ 4

Question 2 (15 marks)

(a) Given the complex number $\omega = \dfrac{5+3i}{2-i}$, find

i) $\overline{\omega}$ **1**

ii) $\omega\overline{\omega}$ **1**

iii) $|\omega|$ **1**

(b) Express $z = \dfrac{\sqrt{2}}{1-i}$ in modulus-argument form and hence find z^5 in the form $x+iy$ **3**

(c) i) Find the five fifth roots of $\sqrt{3}+i$. **2**

ii) Show the five roots of $\sqrt{3}+i$ on an Argand diagram. **2**

(d) Sketch the region in the Argand Plane consisting of those points z for which

$|\arg z| \ngtr \dfrac{\pi}{3}$ intersecting with $|z| \le 3$ **2**

(e) Find the zeros of $P(x) = x^4 - 5x^3 + 7x^2 + 3x - 10$ over the set of complex numbers if $2-i$ is a zero.
Hence factorise *P(x)* completely over the complex set of numbers **3**

Question 3 (15 marks)

Sketch the following curves on separate axes, showing all intercepts and turning points.

(a) $y = x^3 - 4x$ and hence $y = |x^3 - 4x|$ (in the domain: $-3 \le x \le 3$) 4

(b) i) $y = 1 - 2\sin x$ (in the domain: $0 \le x \le 2\pi$) 2

ii) hence $y = |1 - 2\sin x|$ (in the domain: $0 \le x \le 2\pi$) 2

iii) hence $y = \ln|1 - 2\sin x|$ (in the domain: $0 \le x \le 2\pi$) 2

(c) $y = \sqrt{4 - x^2} + 2^x$ (in the domain: $-2 \le x \le 2$) 3

(d) $|y| = 1 - \frac{1}{x}$ (in the domain: $-3 \le x \le 3$) 2

Question 4 (15 marks)

(a) i) For the ellipse $\frac{y^2}{a^2}+\frac{x^2}{b^2}=1$, derive the equation of the tangents at $P(a\cos\theta, b\sin\theta)$ and at the ends of the major axes. 1

ii) Find the coordinates of the points Q and R where the tangent at P meets the two tangents at the extremities of the major axis 2

iii) Hence prove that the interval QR subtends a right angle at either focus 4

(b) $P\left(4p,\frac{4}{p}\right)$ and $Q\left(4q,\frac{4}{q}\right)$ are points on the rectangular hyperbola $xy=16$.

i) Derive the equations of the tangents at P and Q. 2

ii) The tangents at P and Q intersect at the point R.
Derive the coordinates of the point R 2

iii) If the chord PQ passes through the point $(4,0)$, derive the locus of R. 4

Question 5 (15 marks)

(a) Prove by mathematical induction that $\sin(x+n\pi)=(-1)^n\sin x$ where n is a positive integer. 4

(b) Find the general solution to $\sin 4x+\sin 6x=\sin 10x$ 4

Hint: $\sin X+\sin Y = 2\sin\frac{X+Y}{2}\cos\frac{X-Y}{2}$

(c) (i) Differentiate $y=\ln(1+x)$, and hence draw $y=x$ and $y=\ln(1+x)$ on one graph. 2

(ii) Using this graph explain why

$\ln(1+x) < x$, for all $x>0$ 1

(d) (i) Differentiate $\frac{x}{1+x}$, and hence draw $y=\frac{x}{1+x}$ and $y=\ln x$ on one graph. 2

(ii) Using this graph explain why

$\frac{x}{1+x} < \ln(1+x)$ for all $x>0$ 2

Question 6 (15 marks)

(a) The hyperbola H has equation $16x^2 - 9y^2 = 144$

(i) Find the foci points and the equations of the directrices and asymptotes. 3

(ii) Sketch H: $16x^2 - 9y^2 = 144$ 2

(iii) Hence or otherwise find the gradient at the point $(3\sec\vartheta, 4\tan\vartheta)$ 1

(iv) Find the equation of the tangent at $(3\sec\vartheta, 4\tan\vartheta)$ 1

(b) Factorise $P(x)$ over C (the set of complex numbers) if it has a root of multiplicity 3.

$P(x) = 3x^4 + 8x^3 + 6x^2 - 1$ 3

(c) i) Derive the five roots of the equation $z^5 - 1 = 0$ 2

ii) Hence find the exact value of $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5}$ 3

Question 7 (15 marks)

(a) Six letters are chosen from the letters of the word PYTHAGORAS. These six letters are then placed alongside each other to form a six-letter arrangement. Find the number of distinct six-letter arrangements which are possible, considering all the choices. 3

(b) A box contains 6 cards, two of which are identical. From this box 3 cards are drawn without replacement.

(i) How many different selections could be made? 2

(ii) What is the probability that a selection will include the two identical cards 1

(iii) If this process of selecting three cards was repeated, with all cards being replaced after each selection, how many repetitions would be necessary to make the probability of drawing a combination containing the two identical cards at least once, exceed 99%? Hint: Solve for n 3

(c) (i) Give the expansion for $x^3(1+x)^n$ 1

(ii) By differentiating both sides of this binomial expansion, and making a suitable substitution show that

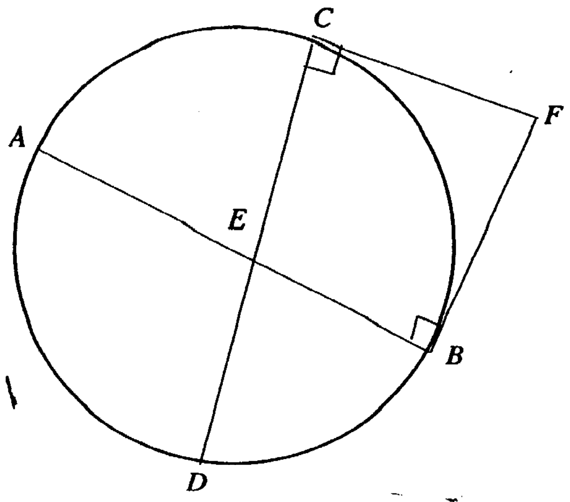
$$2^{n-1}(6+n) = \sum_{r=0}^{n}(r+3).{}^nC_r$$ 3

(d) Solve $3x^2 - 2x - 2 \leq |3x|$ 2

Question 8 (15 marks)

(a) Given that $p>0, q>0, r>0$, prove that $(p+2q)(2q+3r)(3r+p)>48pqr$ 3

(b) In the figure AB and CD are two chords of the circle. AB and CD intersect at E. F is a point such that $\angle ABF$ and $\angle DCF$ are right angles.



Prove that FE produced is perpendicular to AD. 5

(c) Let $f(x) = 3x^5 - 10x^3 + 16x$

(i) Show that $f'(x) \geq 1$ for all x 2

(ii) For what values if x is $f''(x)$ positive? 2

(iii) Sketch the graph of $y = f(x)$ indicating any turning points and points of inflection. 3