Sydney Technical High School

File

Mathematics - Department

TRIAL HSC – MATHEMATICS 2 UNIT

AUGUST 2014

General Instructions

- Reading time – 5 minutes
- Working Time – 180 minutes.
- Approved calculators may be used.
- Write using a blue or black pen.
- A table of Standard Integrals is provided at the back of this paper.
- In Question 11-16, show relevant mathematical reasoning and/or calculations.
- Begin each question on a new side of the answer booklet.
- Marks shown are a guide and may need to be adjusted.
- Full marks may not be awarded for careless work or illegible writing.

NAME____________________

TEACHER____________________

Total marks – 100

SECTION 1

10 Marks

- **Attempt Questions 1 - 10**
- **Allow about 15 minutes.**

SECTION 2

90 marks

- **Attempt Questions 11 - 16**
- **Allow about 2 hours 45 minutes.**

Section 1

Total marks (10)

Attempt Questions 1-10

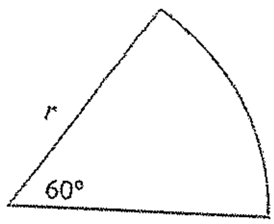
Allow about 15 minutes for this section

Use the multiple-choice answer sheet.
Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

1. Find the values of m for which $24 + 2m - m^2 \leq 0$

   (A) $m \leq -4$ or $m \geq 6$ (B) $m \leq -6$ or $m \geq 4$ (C) $-4 \leq m \leq 6$ (D) $-6 \leq m \leq 4$

2. The sector below has an area of 10π square units.



Not to scale

   What is the value of r?

   (A) $\sqrt{60}$ (B) $\sqrt{60\pi}$ (C) $\sqrt{\frac{\pi}{3}}$ (D) $\sqrt{\frac{1}{3}}$

3. If $\sqrt{7} + \sqrt{28} + \sqrt{63}$ ....$+ p = 300\sqrt{7}$. How many terms are there in the series?

   (A) 24 (B) 300 (C) 298 (D) 25

4. For what values of x is the curve $f(x) = 2x^3 + x^2$ concave down?

   (A) $x < -\frac{1}{6}$ (B) $x > -\frac{1}{6}$ (C) $x < -6$ (D) $x > 6$

5. Given that the curve $y = ax^2 - 8x - 8$ has a stationary point at $x = 2$, find the value of a.

   (A) $a = \frac{1}{2}$ (B) $a = 2$ (C) $a = 6$ (D) $a = -2$

6. If $\int_0^a (4-2x)\ dx = 4$, find the value of a.

(A) $a = -2$ (B) $a = 0$ (C) $a = 4$ (D) $a = 2$

7. If $\tan\ 2x = \sqrt{3}$ in the domain $-\pi \le x \le \pi$, the value of x is:

(A) $\frac{\pi}{6}, \frac{7\pi}{6}$

(B) $-\frac{5\pi}{6}, -\frac{11\pi}{6}$

(C) A and B

(D) None of the above

8. What is the derivative of $\cos^2 3x$ with respect to x?

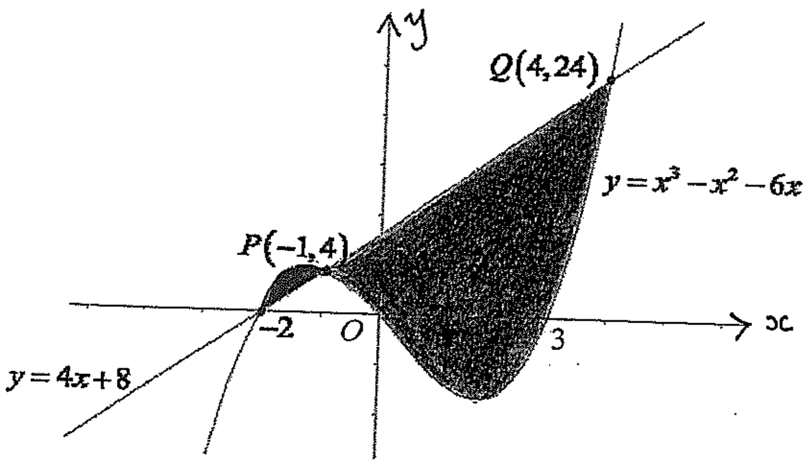
(A) $-2\sin 3x \cos 3x$

(B) $-6\sin 3x \cos 3x$

(C) $2\sin 3x \cos 3x$

(D) $6\sin 3x \cos 3x$

9. Consider the graphs of $y = x^3 - x^2 - 6x$ and $y = 4x + 8$ as illustrated below.



The shaded area enclosed between the curves is given by

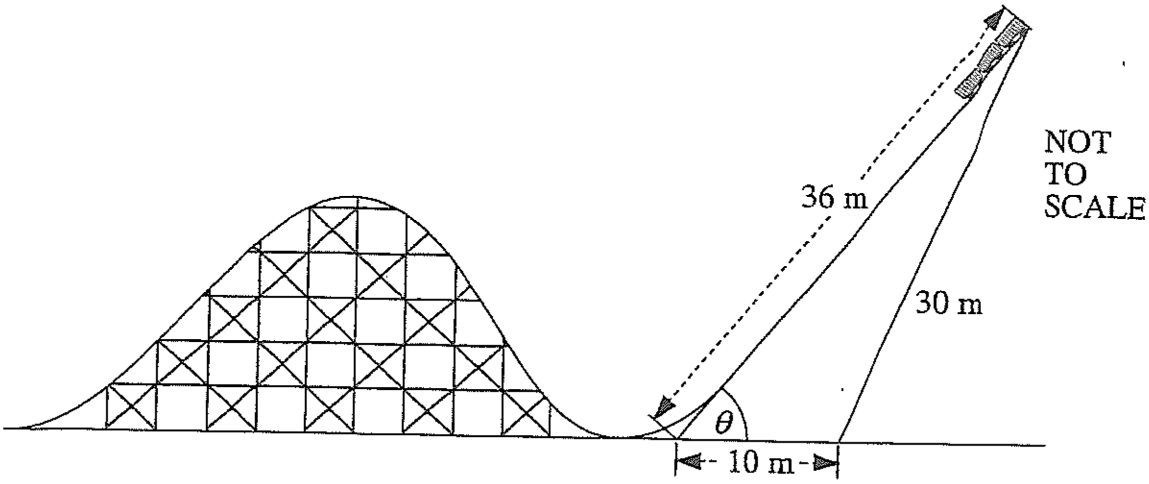
(A) $A = \int_{-2}^{4}(4x+8)dx - \int_{-2}^{4}(x^3 - x^2 - 6x)dx$

(B) $A = \int_{-2}^{4}(x^3 - x^2 - 6x)dx - \int_{-2}^{4}(4x+8\ )dx$

(C) $A = \left|\int_{-2}^{-1}[(4x+8)-(x^3-x^2-6x)]\ dx\right| + \int_{-1}^{4}[(4x+8)-(x^3 - x^2 - 6x)]dx$

(D) $A = \int_{-2}^{-1}[(4x+8)-(x^3-x^2-6x)]\ dx + \left|\int_{-1}^{4}[(4x+8)-(x^3 - x^2 - 6x)]dx\right|$

10.



The diagram shows a fun-park ride. The angle θ is closest to

(A) 46°   (B) 56°   (C) 72°   (D) 74°

Section 2

90 marks

Attempt Question 11 – 16

Allow about 2 hours 45 minutes for this section

Answer each question in the writing book provided. Start each question on a new page.

All necessary working should be shown. Full marks cannot be given for illegible writing.

Question 11 **15 Marks**

a) Evaluate $\sqrt[5]{\frac{1.8+4.2}{3.1-1.6}}$ correct to four significant figures. **2**

b) Factorise $8p^3 + 1$. **2**

c) Solve $|3- 2x| \leq 5$ and graph the solution on a number line. **3**

d) Solve the equation $2\log(x - 5) = \log(2x - 7)$ **2**

e) Evaluate $\sum_{K=1}^{10}(10 - 3k)$. **2**

f) If x, 4 and y are successive terms in an arithmetic sequence and x, 3 and y are successive terms in a geometric sequence, calculate $\frac{1}{x} + \frac{1}{y}$ **2**

g) Consider the function $g(x) = \frac{2}{x^2-1}$

(i) Show that $g(x)$ is an even function. **1**

(ii) State the domain of $y = g(x)$ **1**

Question 12 **Start a new page** **15 marks**

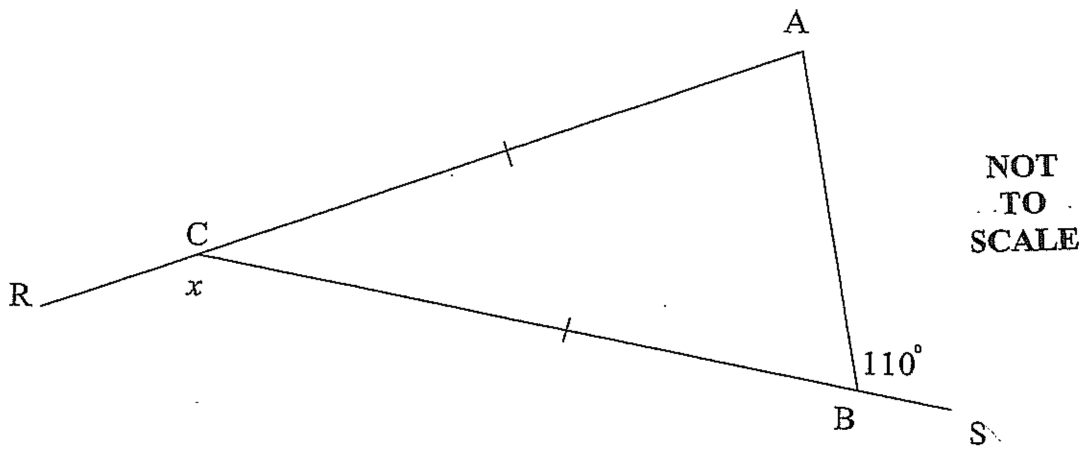
a) Find:

(i) $\int \sec^2 4x dx$ **1**

(ii) $\int \left(\frac{1}{x^2} + \frac{1}{e^{2x}}\right) dx$ **2**

b) Evaluate $\int_0^3 \frac{1}{x+1} dx$ **2**

c)



In the diagram, AC = BC, RCA and CBS are straight lines, ∠ABS = 110° and ∠BCR = x
Copy the diagram onto your writing sheet.
Find the value of x giving reasons **3**

d) Differentiate the following

(i) $y = x^3 \sin x$ **2**

(ii) $y = \sqrt{1 - x^2}$ **2**

(iii) $y = \log_e(1 - x^2)$ **1**

e) Assuming that $v = f(t)$ is a continuous function of time t, with the following set of tabulated values:

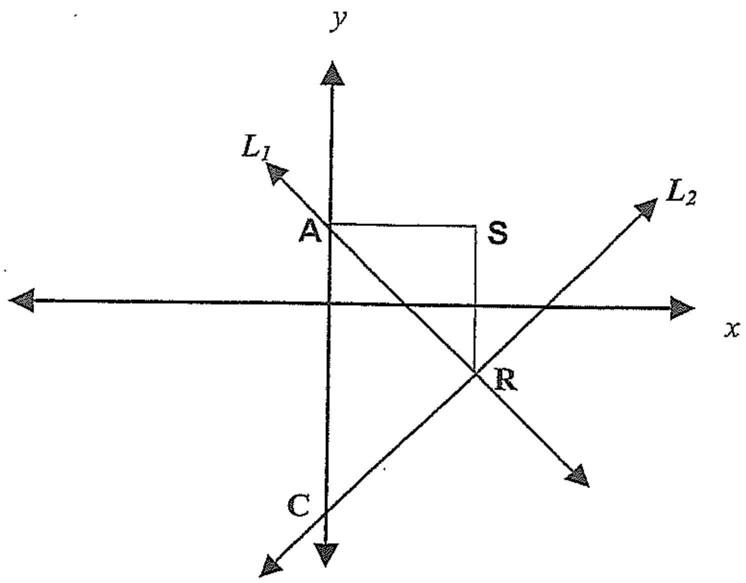
$t(sec)$	0	0.5	1	1.5	2	2.5	3
$v = f(t)(m/sec)$	0	15	32	50	42	30	14

Use Simpson's rule to approximate $\int_0^3 f(t)dt$ **2**

Question 13 Start a new page **15 Marks**

a) Consider the quadratic function $x^2 - (k+2)x + 4 = 0$. **2**
For what value of k does the quadratic function have real roots?

b)



NOT TO SCALE

Line L_1 has equation $x + y = 2$ and intersects the y-axis at point A.
Line L_2 has equation $x - y = 4$ and intersects the y-axis at point C.
Line L_1 and line L_2 intersect at point R.

The horizontal line through A intersects the vertical line through R, at S.

(i) Find the coordinates of point A and C. **2**

(ii) Show that R has coordinates (3, -1). **1**

(iii) State the equation of the line SR **1**

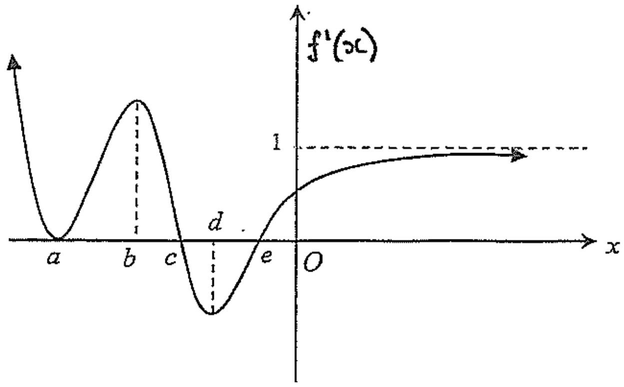
(iv) Find the gradient of line L_1. **1**

(v) Find the distance AR **1**

(vi) Show that triangle ARC is a right-angled isosceles triangle **2**

(vii) Find the equation of the circle with centre R, passing through the points A and C. **1**

c) The graph below represents the gradient function $f\ (x)$.
Specific x values a, b, c, d and e are as indicated in the diagram.



(i) Justify why the graph of $y= f\ (x)$ has a maximum stationary point at $x = c$. **1**

(ii) For what value(s) of x is the graph of $y = f(x)$ increasing and concave up? **2**

(iii) What feature of the graph $y = f(x)$ exists at $x = a$? **1**

Question 14 **Start a new page** **15 marks**

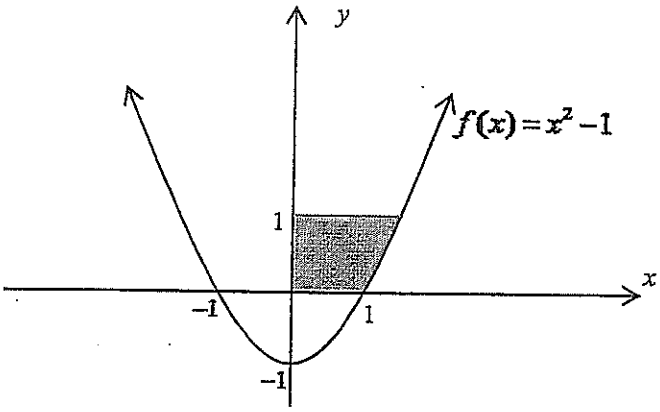
a) Find:

(i) $\int \frac{dx}{\sqrt{3x-2}}$ **2**

(ii) $\int \frac{x^2-3x}{x^3}\,dx$ **2**

b) Evaluate $\int_{-1}^{1} x(x-3)\,dx$ **2**

c) The graph of $f(x) = x^2 - 1$ is shown. The shaded region in the diagram is the area bounded by the curve, the positive x-axis, and the line $y = 1$. Find the volume of the solid of revolution formed when the shaded region is rotated around the y-axis. **2**



d) Consider the curve given by $y = 2x^3 - 3x^2 - 12x$

(i) Find $\frac{dy}{dx}$ **1**

(ii) Find the coordinates of the two stationary points. **2**

(iii) Determine the nature of the stationary points. **2**

(iv) Sketch the curve for $-2 \leq x \leq 3$. Show where the curve cuts the x axis and find the co-ordinates of the end points. **2**

Question 15 **Start a new page** **15 marks**

a) Consider the trigonometric function $y = 1 - 3cos2x$.

(i) State the amplitude of $y = 1 - 3cos2x$. **1**

(ii) Draw a neat and accurate graph of $y = 1 - 3cos2x$ for $0 \leq x \leq \pi$. **2**

(iii) On the same diagram accurately draw the graph of $y = x + 1$. **2**

Hence determine the number of solutions to the equation

$x + 3cos2x = 0$ over the domain $0 \leq x \leq \pi$.

b) A sister city of Sydney is San Francisco. Sydney City Council decides to build an art gallery in San Francisco to allow local Sydney artists to exhibit their work.

The loan required to build the art gallery is $\$P$ with interest charged at the rate of 6% p.a. The loan is to be repaid in equal monthly repayments of \$4000 over 3 years and interest is charged monthly before each repayment.

Let $\$A_n$ be the amount owing by Sydney City Council at the end of the nth repayment

(i) Find an expression for A_1. **1**

(ii) Show that $A_n = P(1.005)^n - 4000(1 + 1.005 + 1.005^2 + \cdots + 1.005^{n-1})$. **2**

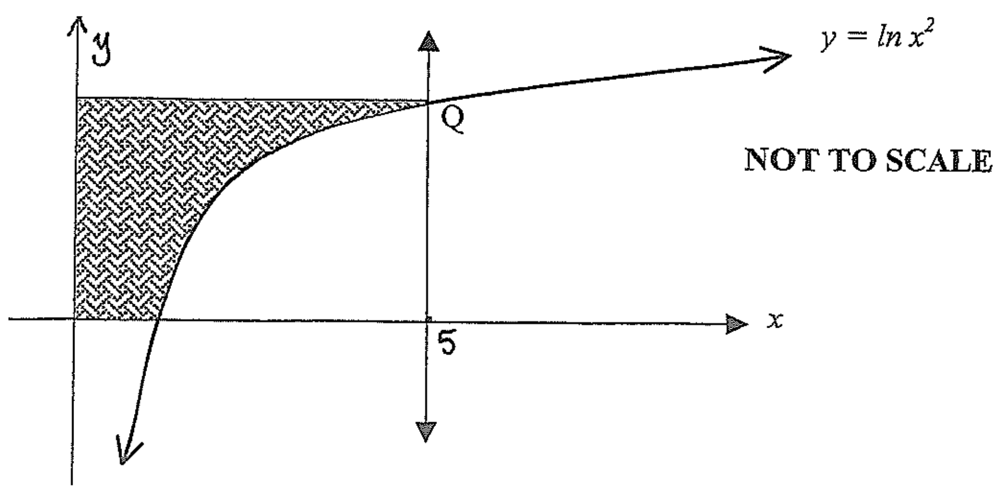
(iii) Hence, find the value of $\$P$. **2**

c)

(i) Show that $\frac{d}{dx}(xlnx - x) = lnx$. **1**

(ii) Hence, or otherwise, find $\int lnx^2 \, dx$. **1**

(iii) The graph shows the curve $y = lnx^2, (x > 0)$ which meets the line $x = 5$ at Q. Using your answers from (i) and (ii), or otherwise, find the area of the shaded region. **3**

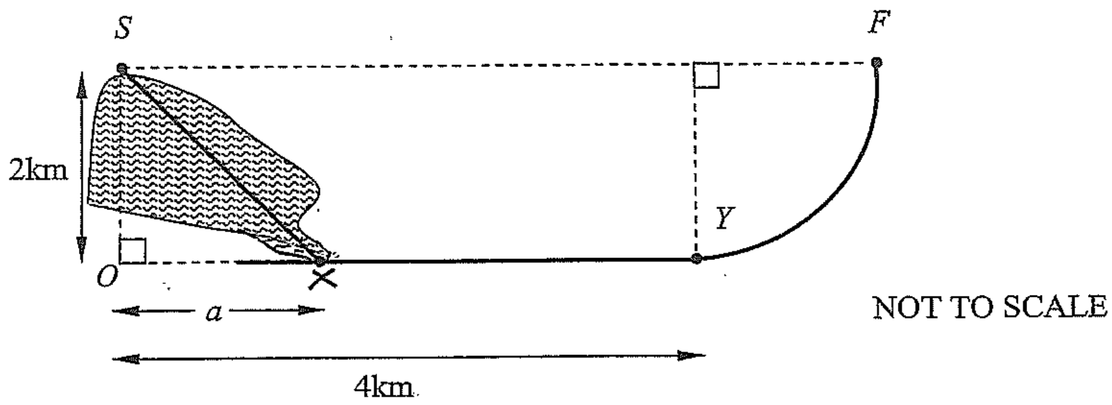


Question 16 Start a new page **15 marks**

a) Find the equation of the parabola with vertex (-1,3) and directrix y=-1. **2**

b) Helen is training to compete in a mini triathlon. The course she practises on consists of three legs which starts at S and finishes at F. The first leg is a straight line swim from S to a point X. The second leg is a bike ride from X to Y along a straight road OY and the final leg is a jog from Y to F around a circular path. The perpendicular distance from S to O is 2km while the distance OY is 4km.

Helen can swim at 6km/h, bike ride at 12km/h and jog at 8km/h.



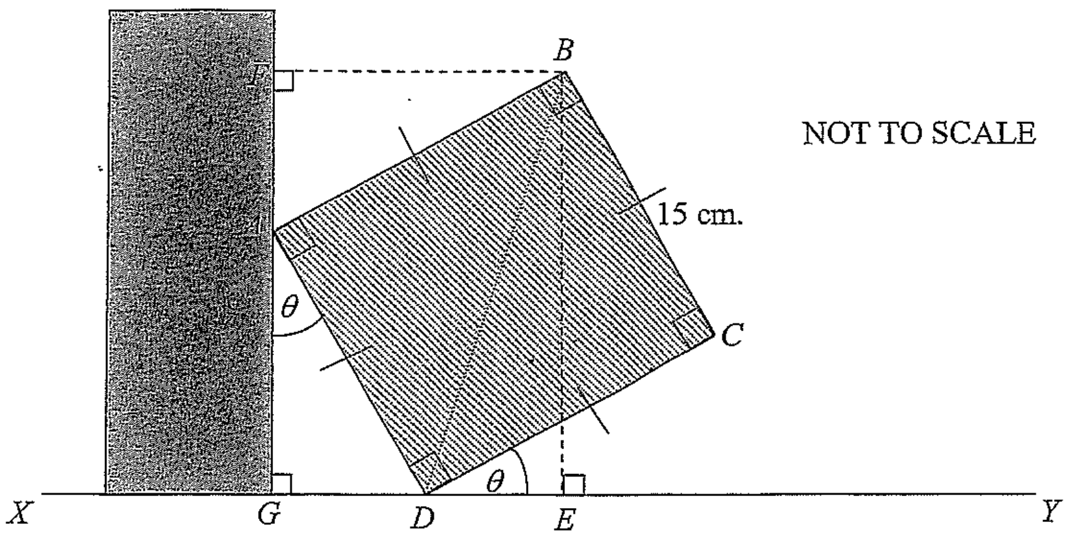
(i) If the distance OX=a km, show that the time T that it takes Helen to complete the three legs is given by **3**

$$T=\frac{4\sqrt{a^2+4}-2a+8+3\pi}{24}\text{ hours.}$$

(ii) Find the value of a, that will allow Helen to minimise the time taken to complete the three legs of her practise course. **3**

(iii) Hence, find the minimum time taken to complete the triathlon to the nearest minute. **1**

c) In the diagram below, *ABCD* is a square of side 15 cm leaning against a wall at an angle θ to the vertical and as well to the ground *XY*.



(i) Show that BD = $15\sqrt{2}$ cm. 1

(ii) Hence by using triangle *DBE*, prove that the perpendicular distance of *B* from the line *XY* is $15\sqrt{2}\,\sin\left(\frac{\pi}{4}+\theta\right)$. 2

(iii) By using triangles *DAG* and *BFA*, find an expression for the length of *FG*. 2

(iv) Hence, prove that $\sin\theta + \cos\theta = \sqrt{2}\,\sin\left(\frac{\pi}{4}+\theta\right)$. 1

End of Exam

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SOLUTIONS

Section 1

1	A
2	A
3	A
4	A
5	B
6	D
7	D
8	B
9	C
10	A

Section 2

Question 11

a) 1.320 (4 sig fig)

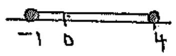
b) $8p^3+1 = (2p+1)(4p^2-2p+1)$

c) $3-2x \leq 5$ $\quad$ $3-2x \geq -5$

$-2x \leq 2$ $\quad$ $-2x \geq -8$

$x \geq -1$ $\quad$ $x \leq 4$

$\therefore -1 \leq x \leq 4$,



d) $2\log(x-5) = \log(2x-7)$

$\log(x-5)^2 = \log(2x-7)$

$(x-5)^2 = 2x-7$

$x^2-10x+25 = 2x-7$

$x^2-12x+32=0$

$(x-8)(x-4)=0$

$\therefore x = 8, 4$

but by subst $2\log(-1)$ no sol.

$\therefore$ $x=8$ only solution

e) $\sum_{k=1}^{10} 10-3k = 7+4+1+\cdots-20$ $\quad$ ($T_1, T_2, \ldots, T_{10}$)

$\therefore S_{10} = \frac{10}{2}(7-20) = -65$

f) $4-x = y-4$ $\quad \therefore x+y=8$

$\frac{x}{3} = \frac{3}{y}$ $\quad xy = 9$

$\frac{1}{x}+\frac{1}{y} = \frac{y+x}{xy}$

$= \frac{8}{9}$

g) i) $g(x) = \frac{2}{x^2-1}$

$g(-x) = \frac{2}{(-x)^2-1} = \frac{2}{x^2-1}$

$g(x) = g(-x)$ $\therefore$ even function

ii) Domain: all real x, $x \neq \pm 1$

Question 12

a) i) $\int \sec^2 4x \, dx$

$= \frac{1}{4}\tan 4x + c$

ii) $\int\left(\frac{1}{x^2} + \frac{1}{e^{2x}}\right)dx$

$\int\left(x^{-2} + e^{-2x}\right)dx$

$= \frac{x^{-1}}{-1} + \frac{e^{-2x}}{-2} + c$

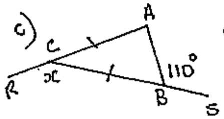
$= -\frac{1}{x} - \frac{1}{2e^{2x}} + c$

b) $\int_0^3 \frac{1}{x+1}\,dx = \left[\ln(x+1)\right]_0^3$

$= \ln 4 - \ln 1$

$= \ln 4$

c)



$\hat{CBA} = 70°$ (angle sum straight line)

$\hat{CAB} = 70°$ (angles opposite equal sides in isosceles triangle)

$\therefore x = 140°$ (exterior angle of ΔABC)

d) i) Let $u = x^3$ $\quad v = \sin x$

$u' = 3x^2$ $\quad v' = \cos x$

$\therefore \frac{dy}{dx} = 3x^2 \sin x + x^3 \cos x$

ii) $y = \sqrt{1-x^2}$

$= (1-x^2)^{1/2}$

$\therefore \frac{dy}{dx} = \frac{1}{2} \times -2x\,(1-x^2)^{-1/2}$

$\therefore \frac{dy}{dx} = \frac{-x}{\sqrt{1-x^2}}$

iii) $y = \log_e(1-x^2)$

$\frac{dy}{dx} = \frac{-2x}{1-x^2}$

e) $\int_0^3 f(t)\,dt = \frac{1/2}{3}\left[0+14+4(15+50+30) + 2(32+42)\right]$

$= \frac{1}{6}(14+380+148)$

$= 90\frac{1}{3}$

Question 13

a) $x^2 - (k+2)x + 4 = 0$

Real Roots $\Delta \geq 0$

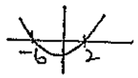
$\Delta = (k+2)^2 - 4.1.4$

$\Delta = k^2+4k+4-16$

$\Delta = k^2+4k-12$

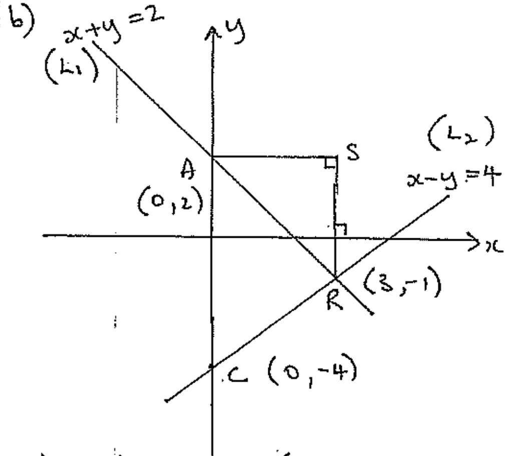
$\therefore k^2+4k-12 \geq 0$

$(k+6)(k-2) \geq 0$



$\therefore k \leq -6, \; k \geq 2$

b)



i) $A(0,2)$ $\quad C(0,-4)$

ii) $x+y=2$
$x-y=4$ $\}+$

$2x = 6$

$x = 3$ $\quad \therefore R(3,-1)$

sub $x=3$ into $x+y=2$

$3+y=2$

$\therefore y=-1$

iii) line SR $\quad x=3$

iv) $x+y=2$

$y = -x+2$

$\therefore m_{L_1} = -1$

v) $AR = \sqrt{(3-0)^2+(-1-2)^2}$

$= \sqrt{9+9}$

$= \sqrt{18}$

$= 3\sqrt{2}$ units

vi) $CR = \sqrt{(3-0)^2+(-1-{-4})^2}$

$= \sqrt{9+9}$

$= \sqrt{18} = 3\sqrt{2}$ $\quad \therefore AR = CR$

gradient $L_1 = -1$

gradient L_2: $x - y = 4$

$x - 4 = y$

is 1

since $m_{L_1} \times m_{L_2} = 1 \times -1 = -1$

$\therefore L_1$ perp to L_2

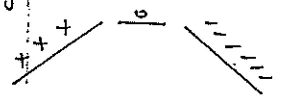
$\therefore \triangle ABC$ is right angled isosceles

vii) $(x-3)^2 + (y+1)^2 = 18$

c) i) gradient to left of c +ve

gradient to right of c −ve

gradient at c is zero



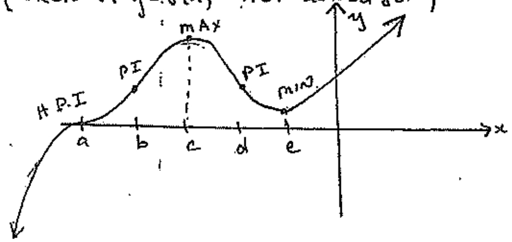
$\therefore$ max s. pt at $x = c$

ii) concave up and increasing

$a < x < b$ & $x > e$

iii) horizontal point of inflexion

(Sketch of $y = f(x)$ not asked for)



Question 14

a) i) $\int (3x-2)^{1/2}\,dx$

$= \dfrac{(3x-2)^{3/2}}{3 \cdot \frac{3}{2}} + c$

$= \dfrac{2\sqrt{3x-2}}{3} + c$

ii) $\int \dfrac{x^2 - 3x}{x^3}\,dx$

$= \int \dfrac{1}{x} - \dfrac{3}{x^2}\,dx$

$= \int x^{-1} - 3x^{-2}\,dx$

$= \ln x - \dfrac{3x^{-1}}{-1} + c$

$= \ln x + \dfrac{3}{x} + c$

b) $\int_{-1}^{1} x^2 - 3x\,dx = \left[\dfrac{x^3}{3} - \dfrac{3x^2}{2}\right]_{-1}^{1}$

$= \left(\frac{1}{3} - \frac{3}{2}\right) - \left(\frac{-1}{3} - \frac{3}{2}\right)$

$= \frac{2}{3}$

c) $V_y = \pi \int_0^1 (y+1)\,dy$

$= \pi \left[\dfrac{y^2}{2} + y\right]_0^1$

$= \pi \left(\frac{1}{2} + 1\right)$

$= \dfrac{3\pi}{2}$ units^3

d) $y = 2x^3 - 3x^2 - 12x$

i) $\dfrac{dy}{dx} = 6x^2 - 6x - 12$

$\dfrac{d^2y}{dx^2} = 12x - 6$

ii) st. pts $\dfrac{dy}{dx} = 0$ $6x^2 - 6x - 12 = 0$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

iii) $x = 2$ $x = -1$ st pts $(2,-20)$ $(-1,7)$

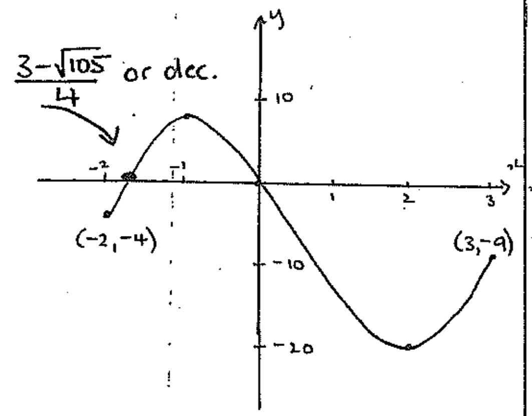
at $(2,-20)$ $y'' > 0$ min

at $(-1, 7)$ $y'' < 0$ max

iv) end pts

$(-2,-4)$

$(3,-9)$

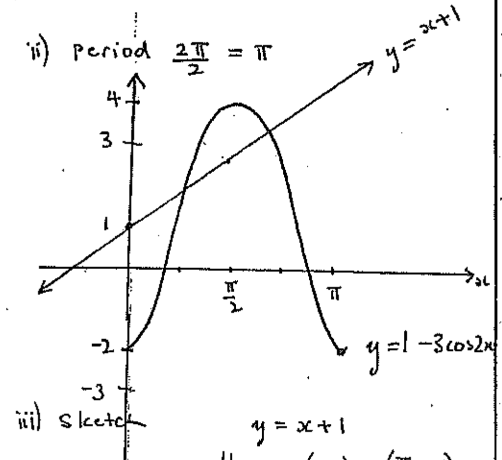


Question 15

a) $y = 1 - 3\cos 2x$

i) amplitude is 3

ii) period $\dfrac{2\pi}{2} = \pi$



iii) sketch

$y = x + 1$

through $(0,1)$ $(\frac{\pi}{2}, 2.6)$

iv) $1 - 3\cos 2x = x + 1$ sim. eq

$0 = x + 3\cos 2x$

has 2 solutions in domain

$0 \leq x \leq \pi$

b) 6% p.a = 0.5% pm

i) $A_1 = P\left(1 + \frac{.5}{100}\right)^1 - 4000$

$A_1 = P(1.005)^1 - 4000$

ii) $A_2 = \left[P(1.005)^1 - 4000\right](1.005)^1 - 4000$

$= P(1.005)^2 - 4000(1.005)^1 - 4000$

$\therefore A_3 = P(1.005)^3 - 4000(1.005)^2 - 4000(1.005) - 4000$

$\therefore A_3 = P(1.005)^3 - 4000(1 + 1.005^1 + 1.005^2)$

↑ GP $a=1$ $r=1.005$ $n=3$

$\therefore$

$A_n = P(1.005)^n - 4000(1 + 1.005^1 + \cdots + 1.005^{n-1})$

iii) $A_n = 0$ loan repaid

$P(1.005)^n = 4000(1 + 1.005^1 + \cdots 1.005^{n-1})$

$n = 36$

$\therefore P(1.005)^{36} = 4000(1 + 1.005^1 + \cdots + 1.005^{35})$

↑ GP $a=1$ $r=1.005$ $n=3$

$P = \dfrac{4000(1.005^{36} - 1)}{(1.005)^{36}(1.005 - 1)}$

$P = \$131,484.07$

c) $u = x$ $v = \ln x$

i) $u' = 1$ $v' = \dfrac{1}{x}$

$\dfrac{d}{dx}(x\ln x - x) = (\ln x - x.\frac{1}{x} - 1)$

$\dfrac{d}{dx}(x\ln x - x) = \ln x$

ii) $\int \ln x^2\,dx = \int 2\ln x\,dx$

$= 2\int \ln x\,dx$

$= 2\left[x\ln x - x\right] + c$

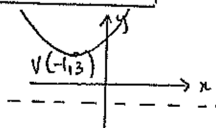
iii) $A_x = $ Rect $(5 \times \ln 25) - \int_1^5 \ln x^2\,dx$

$= 5\ln 25 - \left[2(x\ln x - x)\right]_1^5$

$A = 5\ln 5^2 - [(10\ln 5 - 10) - (2\cancel{\ln 1} - 2)]$
$= 10\ln 5 - [10\ln 5 - 10 + 2]$
$= 10\ln 5 - 10\ln 5 + 8$
$= 8 \text{ unit}^2$

Question 16

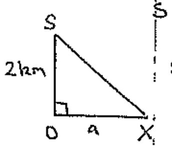
a)



$a = 4$

parab $(x+1)^2 = 4 \cdot a\,(y-3)$
$(x+1)^2 = 16\,(y-3)$

b) $D = ST$
$\therefore T = \frac{D}{S}$



$SX = \sqrt{a^2+4}$ speed 6km/h
$\therefore T_{SX} = \frac{\sqrt{a^2+4}}{6}$

$XY = 4 - a$ speed 12km/h
$T_{XY} = \frac{4-a}{12}$

$FY = \frac{2\pi 2}{4} = \pi$ speed 8km/h

$\therefore T_{FY} = \frac{\pi}{8}$

$\therefore$ Total time $= \frac{\sqrt{a^2+4}}{6} + \frac{(4-a)}{12} + \frac{\pi}{8}$
$= \frac{4\sqrt{a^2+4} + 2(4-a) + 3\pi}{24}$
$T = \frac{4\sqrt{a^2+4} + 8 - 2a + 3\pi}{24}$

ii) $T = \frac{1}{6}(a^2+4)^{1/2} - \frac{a}{12} + \frac{1}{3} + \frac{\pi}{8}$

$\frac{dT}{da} = \frac{1}{6}\cdot 2a \cdot \frac{1}{2}(a^2+4)^{-1/2} - \frac{1}{12}$

st pt $\frac{dT}{da} = 0$

$\frac{a}{6\sqrt{a^2+4}} = \frac{1}{12}$
$12a = 6\sqrt{a^2+4}$
$2a = \sqrt{a^2+4}$
$4a^2 = a^2 + 4$
$3a^2 = 4$
$a^2 = 4/3$
$a = \pm\frac{2}{\sqrt{3}}$
$a > 0$
test $2/\sqrt{3}$

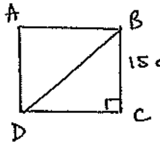
a	1	$2/\sqrt{3}$	2
T'	−	0	+

$\therefore$ min

$\therefore$ min time if $a = \frac{2}{\sqrt{3}}$ km

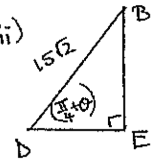
iii) min Time = 61 mins.

c) i)



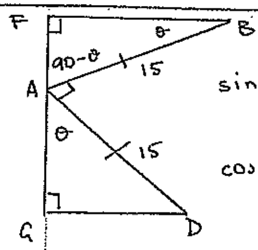
$BD^2 = 2 \times 15^2$
$BD = 15\sqrt{2}$ cm

ii)



$\sin\left(\frac{\pi}{4}+\theta\right) = \frac{BE}{15\sqrt{2}}$
$BE = 15\sqrt{2}\sin\left(\frac{\pi}{4}+\theta\right)$

iii)



$\sin\theta = \frac{FA}{15}$
$\cos\theta = \frac{AG}{15}$

$\therefore FA = 15\sin\theta$
$AG = 15\cos\theta$
$\therefore FG = 15\cos\theta + 15\sin\theta$

iv) $BE = FG$
$\cancel{15}\sqrt{2}\sin\left(\frac{\pi}{4}+\theta\right) = \cancel{15}(\cos\theta + \sin\theta)$

$\sqrt{2}\sin\left(\frac{\pi}{4}+\theta\right) = \cos\theta + \sin\theta$