

2014
HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 1

General Instructions

- Reading Time - 5 minutes
- Working Time - 2 hours
- Write using a blue or black pen
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- Show all necessary working in Questions 11 - 14
- Begin each question on a new sheet of paper.

Total marks (70)

Section I

10 marks

- Attempt Questions 1-10
- Answer on the Multiple Choice answer sheet provided.
- Allow about 15 minutes for this section

Section II

60 marks

- Attempt questions 11 – 14
- Answer on the blank paper provided, unless otherwise instructed. Start a new page for each question.
- Allow about 1 hour 45 minutes for this section

Section I

Total marks (10)

Attempt Questions 1-10

Allow about 15 minutes for this section

Use the multiple choice answer sheet.

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample

$2 + 4 = ?$ (A) 2 (B) 6 (C) 8 (D) 9

A B C D

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A B C D

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows:

A B C D

correct ↗

Question 1 Which of the following is an expression for $\int \cos^2 x \sin x dx$?

(A) $2 \cos x \sin x + c$ (B) $\cos^3 x + c$

(C) $-\frac{1}{3} \cos^3 x + c$ (D) $\frac{1}{3} \cos^3 x + c$

Question 2 A particle is moving along the x -axis. Its velocity v at position x is given by $v = \sqrt{8x - x^2}$. What is the acceleration when $x = 3$?

(A) 1 (B) 2

(C) 3 (D) 4

Question 3 If $f(x) = e^{x+2}$ what is the inverse function $f^{-1}(x)$?

(A) $f^{-1}(x) = e^{x-2}$ (B) $f^{-1}(x) = e^{x+2}$

(C) $f^{-1}(x) = \log_e x - 2$ (D) $f^{-1}(x) = \log_e x + 2$

Question 4 A particle is moving in a straight line with $v^2 = 36 - 4x^2$ and undergoing simple harmonic motion. If the particle is initially at the origin, which of the following is the correct equation for its displacement in terms of t ?

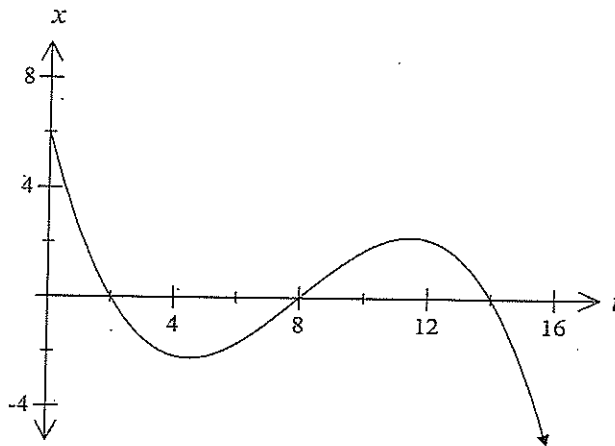
(A) $x = 2 \sin(3t)$ (B) $x = 3 \sin(2t)$

(C) $x = 2 \sin(9t)$ (D) $x = 3 \sin(4t)$

Question 5 What is the solution to the equation $\frac{\cos^3 \theta}{\sin \theta} + \sin \theta \cos \theta = 1$ in the domain $0 \leq \theta \leq 2\pi$?

- (A) $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$ (B) $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$
 (C) $\theta = \frac{5\pi}{4}, \frac{7\pi}{4}$ (D) $\theta = \frac{3\pi}{4}, \frac{5\pi}{4}$

Question 6 The displacement, x metres, from the origin of a particle moving in a straight line at any time (t seconds) is shown in the graph.



When was the particle at rest?

- (A) $t = 4.5$ and $t = 11.5$ (B) $t = 0$
 (C) $t = 2, t = 8$ and $t = 14$ (D) $t = 1.5$ and $t = 8$

Question 7 The population (P) of a colony of bugs is increasing continuously at a rate proportional to the existing population. The present population is 20 000 and the population 3 months ago was 8000. If $P = Ae^{kt}$, what is the value of k ?

- (A) -0.916 (B) -0.305
 (C) 0.305 (D) 0.916

Question 8 A particle moves along a straight line about a fixed point O so that its acceleration, $a \text{ ms}^{-2}$, at time t seconds is given by $a = 4 \cos\left(2t + \frac{\pi}{6}\right)$. Initially the particle is moving to the right with a velocity of 1 ms^{-1} from a position $\frac{\sqrt{3}}{2}$ metres to the left of O . Which of the following is the correct expression for the velocity of the particle after t seconds?

- (A) $v = 2 \sin\left(2t + \frac{\pi}{6}\right)$ (B) $v = 2 \sin\left(2t + \frac{\pi}{6}\right) + 1 - \sqrt{3}$
- (C) $v = 4 \sin\left(2t + \frac{\pi}{6}\right)$ (D) $v = 4 \sin\left(2t + \frac{\pi}{6}\right) - 1$

Question 9 What is the exact value of the definite integral $\int_{\frac{2}{\sqrt{3}}}^{2\sqrt{3}} \frac{dx}{x^2 + 4}$?

- (A) $\frac{\pi}{12}$ (B) $\frac{\pi}{6}$
- (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

Question 10 A particle moves in a straight line and its position at any time t is given by $x = 3 \cos 2t + 4 \sin 2t$. The motion is simple harmonic. What is the greatest speed?

- (A) 6 (B) 10
- (C) 12 (D) 20

End of Section 1

Section II

Total marks (60)

Attempt Questions 11 - 14

Allow about 1 hour 45 minutes for this section.

Answer all questions, starting each question on a new page and question number at the top of the page.

Question 11 (15 marks) Use a separate page/booklet	Marks
(a) Find $\lim_{x \rightarrow \infty} \frac{3x^2 - 2x}{x^2 + 4}$	1
(b) (i) Show that $x - 2$ is a factor of $x^3 - 4x^2 + 7x - 6$	1
(ii) Explain why $x^3 - 4x^2 + 7x - 6 = 0$ has only 1 real root.	2
(c) Find the value of k if the roots of the equation $x^3 - 3x^2 - 6x + k = 0$ are in arithmetic progression.	3
(d) Differentiate with respect to x : $y = \log_7 x^2$	2
(e) Find the coordinates of the point P that divides the interval $(2, -6)$ and $(7, 9)$ internally in the ratio $2 : 3$.	2
(f) The acceleration of a particle is given by $a = -e^{-x}$. Initially $v = \sqrt{2}, x = 0$. Find the velocity as a function of x .	2
(g) (i) State the domain of $y = 4 \cos^{-1} \frac{x}{3}$	1
(ii) Hence, sketch the curve $y = 4 \cos^{-1} \frac{x}{3}$	1

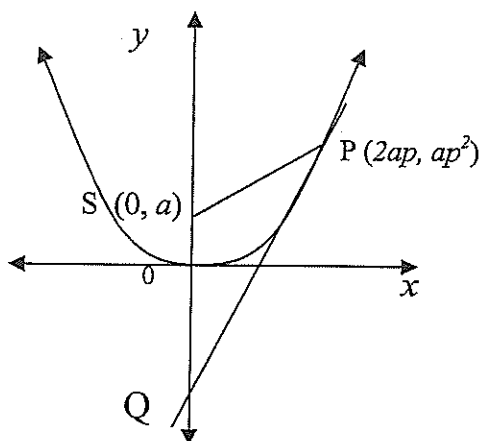
Question 12 (15 marks) Use a separate page/booklet

Marks

(a) Solve: $\frac{x}{x-2} \geq 4, \quad x \neq 2$

2

(b) The point $P(2ap, ap^2)$ lies on the parabola $x^2 = 4ay$ whose focus is at S. The tangent at P meets the Y-axis at Q



(i) Derive and show that the equation of the tangent at P is $y = px - ap^2$.

2

(ii) Find the coordinates of Q.

1

(ii) Show that $\angle SPQ = \angle SQP$

2

(c) A particle moves in simple harmonic motion. It starts from rest at a point 6 cm to the right of the centre of motion O. The particle has a speed of 10 cm/s, when it passes through O.

(i) Write the expression for displacement in the form of $x = a \cos(nt + \alpha)$.

2

(ii) Find the period of motion.

1

(iii) Find the acceleration after 3 seconds.

1

(d) (i) Express $\sqrt{3} \sin 2\theta - \cos 2\theta$ in the form $R \sin(2\theta - \alpha)$, where α acute.

2

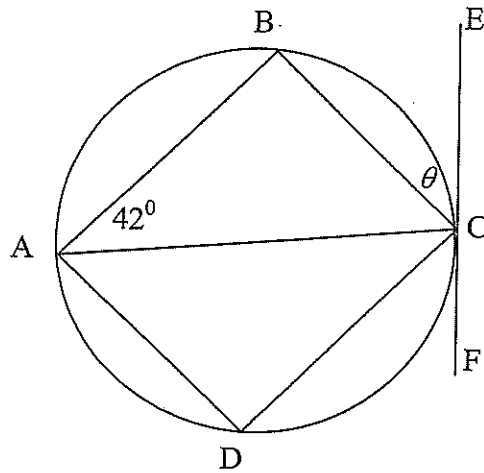
(ii) Hence solve $\sqrt{3} \sin 2\theta - \cos 2\theta = 1 ; 0 \leq \theta \leq \pi$. Answer in exact form.

2

Question 13 (15 marks) Use a separate page/booklet.

Marks

- (a) ABCD is a cyclic quadrilateral where AC bisects $\angle DAB$, $\angle BAC = 42^\circ$ and FE is a tangent to the circle at C.

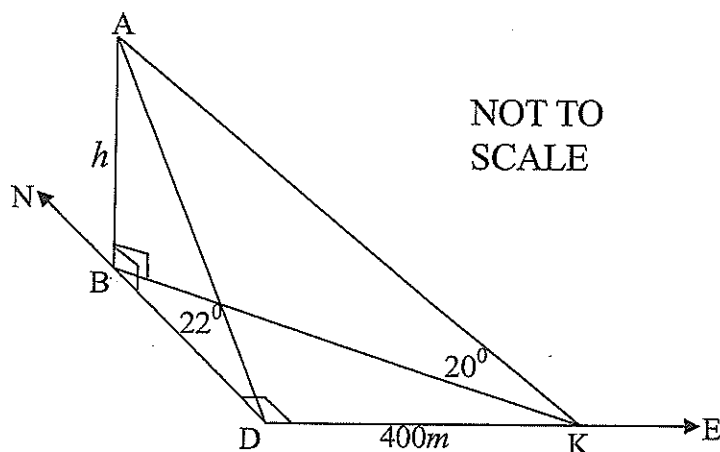


NOT TO SCALE

- (i) Find the size of θ ($\angle BCE$). Give reasons. 1
- (ii) Prove that FE is parallel to DB . 3
- (b) (i) Differentiate: $y = x \sin^{-1} x + \sqrt{1-x^2}$ 2
- (ii) Hence evaluate $\int_0^{\frac{1}{2}} \sin^{-1} x \, dx$ 1
- (c) Prove that $\tan^{-1} x = \sin^{-1} \frac{x}{\sqrt{1+x^2}}$ 2

Question 13 continues on page 8

- (d) Donna is standing at D and observes the angle of elevation of the tip of a flagpole A, on top of a building to be 22° . Her friend Kate, who is standing at K, 400 metres due east of Donna, finds the angle of elevation of the tip of the flagpole to be 20° . The building is due north of Donna and B is the base of the building. The points B, D and K are all on level ground.



- (i) Show that the height (h) of the flagpole above the ground is given by:

$$h = \frac{400}{\sqrt{\cot^2 20^\circ - \cot^2 22^\circ}}$$

3

- (ii) Find the value of h , correct to 3 significant figures.

1

- (e) Using the substitution $x = u^2 - 2$ or otherwise, find $\int \frac{x}{\sqrt{x+2}} dx$

2

Question 14 (15 marks) Use a separate page/booklet.

Marks

(a) A particle is projected with speed v m/s at an angle of projection, θ to the horizontal.

(i) Derive expressions for the horizontal and vertical displacements x and y at any time t seconds after projection. Let gravity = g m/s².

2

(ii) Show that the equation of the path of the particle is given by

$$y = x \tan \theta - \frac{gx^2}{2v^2 \cos^2 \theta}$$

2

(iii) The particle has an initial speed of $2\sqrt{70}$ m/s and just clears a pole. The pole is 5m high and its base is 20m from the point of projection. Find two possible angles of projection to the nearest degree. (Take $g = 9.8$ m/s²)

2

(b) Prove by Mathematical Induction that,

3

$(n)^3 + (n+1)^3 + (n+2)^3$ is divisible by 9 for all positive whole numbers n

(c) (i) Sketch the curve $y = \ln(x-2)$

1

(ii) The inner surface of a bowl is of the shape formed by rotating about the y axis, the curve $y = \ln(x-2)$ between $y = 0$ and $y = 2$

The bowl is placed with its axis vertical and water is poured in.

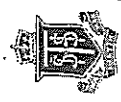
Show that the volume of water in the bowl when it is filled to a depth

h , where $h < 2$, is given by $\pi(4h - 4\frac{1}{2} + 4e^h + \frac{1}{2}e^{2h})$ unit³.

3

(iii) If the bowl is filled at the rate of 60 unit³ / s, find the rate at which the water level is rising when the depth of water is 1.25 units. Give your answer correct to 2 decimal places.

2



MC: C A C B B A C A A B

Q11

(a) $\lim_{x \rightarrow \infty} \frac{3x^2 - 2x}{x}$

$\lim_{x \rightarrow \infty} \frac{x^2}{x} - \frac{2x}{x}$

$= \lim_{x \rightarrow \infty} x - \frac{2}{x}$

$= \lim_{x \rightarrow \infty} x - \frac{4}{2x}$

$= \lim_{x \rightarrow \infty} x = 3$

$\lim_{x \rightarrow \infty} \frac{3x^2 - 4x^2 + 7x - 6}{x^2 - 2x + 3} = -6$

(b)(i) $f(x) = x^3 - 4x^2 + 7x - 6$

$f(2) = 2^3 - 4(2)^2 + 7(2) - 6 = 0$

(ii) $x^2 - 2x + 3$

$x^2 - 4x^2 + 7x - 6 = 0$

$x^3 - 2x^2 - 2x^2 + 7x - 6$

$-2x^2 + 7x - 6$

$-2x^2 + 4x + 3x - 6$

$3x - 6$

$\therefore (x-2)(x^2 - 2x + 3) = 0$

$x^2 - 2x + 3 = 0$ is a factor and has no solution

$x = 2 \pm \sqrt{4 - 4 \times 1 \times 3}$

$= \frac{2 \pm \sqrt{-8}}{2}$

\therefore no solution

①

(c) let the roots be $x, \alpha+d, \alpha+d$

$x + \alpha - d + \alpha + d = -3$

$3\alpha = 3$

$\alpha = 1$

$x(\alpha-d) + x(\alpha+d) + (\alpha-d)(\alpha+d) = -6$

$x^2 - \alpha d + \alpha^2 + \alpha d + \alpha^2 - d^2 = -6$

$\therefore 3\alpha^2 - d^2 = -6$

$3 - d^2 = -6$

$d^2 = 9$

$x(\alpha-d)(\alpha-d) = -4$

$x(d^2 - d^2) = -4$

$1(1^2 - d^2) = -4$

$-8 = -4$

$\therefore 4 = 8$

(d) $y = \frac{\ln x^2}{\ln 7}$

$= \frac{2}{\ln 7} \cdot \ln x$

$y' = \frac{2}{\ln 7} \cdot \frac{1}{x}$

$= \frac{2}{x \ln 7}$

$= \frac{2}{x \ln 7}$

$= \frac{2}{x \ln 7}$

$= \frac{2}{x \ln 7}$

②



(e) $(2x^7 + 3x^2) \div (2x^3 + 3)$

$= (4, 0)$

(f) $a = e^{-2x}$

$\frac{d(\frac{1}{2}v^2)}{dx} = -e^{-2x}$

$\frac{1}{2}v^2 = \int -e^{-2x}$

$= e^{-2x} + c$

$\frac{1}{2}(\sqrt{x})^2 = e^{-0} + c$

$1 = 1 + c$

$c = 0$

$\frac{1}{2}v^2 = e^{-2x}$

$v^2 = 2e^{-2x}$

$v = \sqrt{2}e^{-x}$

(g)(i)

$y - ap^2 = -2ap^2$

$y - ap^2 = -2ap^2$

$y = -ap^2$

$\frac{d^2 y}{dx^2} = (2ap)^2 + (ap^2 - a)^2$

$= 4a^2 p^4 + a^2 p^4 - 2a^2 p^2 + a^2$

$= a^2 p^4 + 2a^2 p^2 + a^2$

$= a^2 (p^4 + 2p^2 + 1)$

$= a^2 (p^2 + 1)^2$

②

(d) $\frac{2x(x-2)}{x-2} > 4(x-2)^2$

$x(x-2) > 4(x-2)^2$

$4(x-2)^2 - x(x-2) \leq 0$

$(x-2)[4x-8-x] \leq 0$

$(x-2)(3x-8) \leq 0$

$\therefore 2 < x \leq \frac{8}{3}$

(h)(i) $y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}$

$y - ap^2 = p(x-2ap)$

$y - ap^2 = p(x-2ap)$

$y = -ap^2$

$\frac{d^2 y}{dx^2} = (2ap)^2 + (ap^2 - a)^2$

$= 4a^2 p^4 + a^2 p^4 - 2a^2 p^2 + a^2$

$= a^2 p^4 + 2a^2 p^2 + a^2$

$= a^2 (p^4 + 2p^2 + 1)$

$= a^2 (p^2 + 1)^2$

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②



$$d_{sq} = \sqrt{a^2(\rho^2 + 1)}$$

$$= a(\rho^2 + 1) = a\rho^2 + a$$

$$d_{sq} = a + a\rho^2$$

$$= d_{sq}$$

\therefore LSPA = LSAQ (opposite sides)

equal \therefore opposite equals sides (ii) period = $\frac{2\pi}{\frac{2\pi}{3}} = 3$

(C) (i) $x = a \cos(ut + \alpha)$

when $t=0, x=6$

$$6 = a \cos \alpha$$

$$\therefore -a \sin(ut + \alpha)$$

when $t=0, \dot{x}=0$

$$0 = -a \sin \alpha$$

$$\sin \alpha = 0 \therefore \alpha = 0$$

$$\therefore \cos \alpha = 1$$

$$\therefore a = 6 \quad x = 6 \cos ut$$

when $x=0, \sin(ut) = 0 \therefore \cos ut = 0$

$$\therefore \cos ut = 0$$

$$ut = \frac{\pi}{2}, \frac{3\pi}{2}, \dots$$

$$\frac{\pi}{2}, 2$$

$$t = \frac{\pi}{2u}, \frac{3\pi}{2u}, \dots$$

$$|-6 \sin ut| = 10$$

$$6 \sin ut = \frac{10}{2u} = 10$$

(3)

$$\therefore \theta = \frac{\pi}{3}, \frac{2\pi}{3}$$



a) i) $\theta = 42^\circ$ (angle in obtuse seg.)

ii) $\angle CAD = 42^\circ$ ($\angle BAD$ is bisected)

$\therefore \angle OCF = 42^\circ$ (\angle in obt. seg.)

Given DB

$\therefore \angle OBC = 42^\circ$ (alt. angles)

\therefore BD || FE (alt. angles equal)

(b) i) $y = r \cos \theta + \sin^{-1} x + \sqrt{1-x^2}$

$$= \frac{x}{\sqrt{1-x^2}} + \sin^{-1} x - x$$

$$= \sin^{-1} x$$

$$= \frac{x}{\sqrt{1-x^2}}$$

$$= \frac{1}{\sqrt{1-x^2}}$$

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d) i) $\frac{h}{BK} = \tan 22^\circ$

$$BK = \frac{h}{\tan 22^\circ}$$

$$BD = \frac{h}{\tan 22^\circ}$$

$$BD^2 + 400^2 = BK^2$$

$$h^2 \cot^2 22^\circ - h^2 (\cot^2 22^\circ - 400^2) = 400^2$$

$$h^2 (\cot^2 22^\circ - \cot^2 22^\circ + 400^2) = 400^2$$

$$h^2 = 400^2$$

$$h = 400$$

$$h = 335 \text{ metres}$$

$$r = u^2 - 2$$

$$\frac{dr}{dt} = 2u \frac{du}{dt}$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{4} - 1 = \frac{\pi}{12} + \frac{\sqrt{3}}{4} - 1$$

$$\int \frac{r}{\sqrt{r+2}} \cdot dr = \int \frac{u^2 - 2}{\sqrt{u^2 - 2}} \cdot 2u \cdot du$$

$$= 2 \int \sqrt{u^2 - 2} \cdot du$$

$$= \frac{2u^3}{3} - 4u + C$$

$$= 2 \frac{(20+2)^{3/2}}{3} - 4(20+2)^{1/2} + C$$

$$= \frac{2(22)^{3/2}}{3} - 4(22)^{1/2} + C$$

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Q14 a) i)

$$\dot{x} = 0 \quad \dot{y} = -g$$

$$x = c \quad y = -gt + c$$

$$\dot{x} = v \cos \theta \quad \dot{y} = -gt + v \sin \theta$$

$$x = vt \cos \theta + c \quad y = -\frac{1}{2}gt^2 + vt \sin \theta$$

when $t=0, x=c, y=0$

∴ $x = \sqrt{t \cos \theta}$ & $y = -\frac{1}{2}gt^2 + vt \sin \theta$

ii) $x = \frac{v}{\cos \theta}$

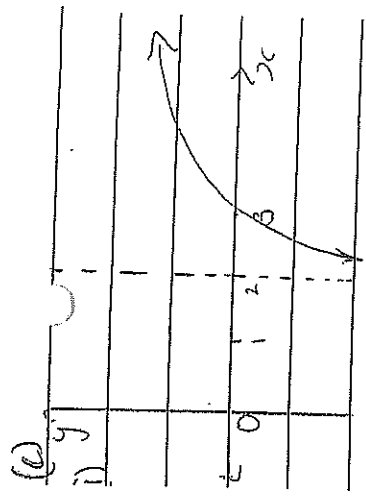
$$y = \frac{-\frac{1}{2}gt^2}{2\sqrt{t \cos \theta}} + \frac{v \sin \theta}{\sqrt{t \cos \theta}}$$

$$= -\frac{g}{2\sqrt{t \cos \theta}} + x \tan \theta$$

$$= x \tan \theta - \frac{g}{2\sqrt{t \cos \theta}}$$

iii) $v = 2\sqrt{10}, x = 20, y = 5,$
 $g = 9.8$
 $20 \tan \theta - \frac{7}{\cos \theta} = 5$
 $20 \tan \theta - 7 \sec^2 \theta = 5$
 $20 \tan \theta - 7(1 + \tan^2 \theta) = 5$
 $20 \tan \theta - 7 - 7 \tan^2 \theta = 5$
 $7 \tan^2 \theta - 20 \tan \theta + 12 = 0$

(5)



(i) $V = \pi \int_0^2 x^2 \cdot dy$

$$\ln(6x-2) = y$$

$$6x-2 = e^y$$

$$x = \frac{e^y + 2}{6}$$

$$x^2 = \left(\frac{e^y + 2}{6}\right)^2$$

$$= \frac{e^{2y} + 4e^y + 4}{36}$$

$$\therefore V = \pi \int_0^2 \frac{e^{2y} + 4e^y + 4}{36} \cdot dy$$

$$= \frac{\pi}{36} \left[\frac{e^{2y}}{2} + 4e^y + 4y \right]_0^2$$

$$= \frac{\pi}{36} \left[\frac{1}{2}(e^4 + 4e^2 + 8) - \left(\frac{1}{2} + 4\right) \right]$$

$$= \frac{\pi}{36} \left[\frac{1}{2}e^4 + 2e^2 + 4 - \frac{9}{2} \right]$$

ii) $\frac{dV}{dt} = \frac{dV}{dn} \times \frac{dn}{dt}$

$$60 = \pi (e^{2n} + 4e^n + 4) \times \frac{dn}{dt}$$

$$\frac{dn}{dt} = \frac{60}{\pi (e^{2 \times 1.25} + 4e^{1.25} + 4)}$$

≈ 0.63

(6)