SYDNEY GRAMMAR SCHOOL

2009 Trial Examination

FORM VI
MATHEMATICS EXTENSION 2

Tuesday 11th August 2009

General Instructions
- Reading time — 5 minutes
- Writing time — 3 hours
- Write using black or blue pen.
- Board-approved calculators may be used.
- A list of standard integrals is provided at the end of the examination paper.
- All necessary working should be shown in every question.
- Start each question in a new booklet.

Structure of the paper
- Total marks — 120
- All eight questions may be attempted.
- All eight questions are of equal value.

Collection
- Write your candidate number clearly on each booklet and on the tear-off sheet.
- Hand in the eight questions in a single well-ordered pile.
- Hand in a booklet for each question, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Bundle the tear-off sheet with the question it belongs to.
- Bundle the tear-off sheet with the question it belongs to.
- Place the question paper inside your answer booklet for Question 1.

Checklist
- SGS booklets — 8 per boy
- Candidature — 72 boys

Examiner
DNW

QUESTION ONE (15 marks) Use a separate writing booklet. **Marks**

(a) Evaluate $\displaystyle\int_0^1 xe^{x^2}\,dx$. **2**

(b) Complete the square to find $\displaystyle\int \frac{dx}{x^2-2x+5}$. **2**

(c) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} x\sin x\,dx$. **3**

(d) (i) Find values of a, b and c such that **3**

$$\frac{x^2+2x-4}{(x+1)(x^2+4)} = \frac{a}{x+1} + \frac{bx+c}{x^2+4}\,.$$

(ii) Hence evaluate $\displaystyle\int_0^1 \frac{x^2+2x-4}{(x+1)(x^2+4)}\,dx$. **3**

(e) Use the substitution $x = \frac{\pi}{2} - u$ to show that **2**

$$\int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1+\sin 2x}\,dx = 0\,.$$

Exam continues next page ...

QUESTION TWO (15 marks) Use a separate writing booklet. **Marks**

(a) Let $z = 3 - 4i$ and $w = 2 + i$. Find, in the form $x + iy$:

(i) $z + iw$ **1**

(ii) $z\overline{w}$ **1**

(b) Let $\alpha = 1 - i$.

(i) Write α in modulus-argument form. **1**

(ii) Hence show that $\alpha^4 + 4 = 0$. **2**

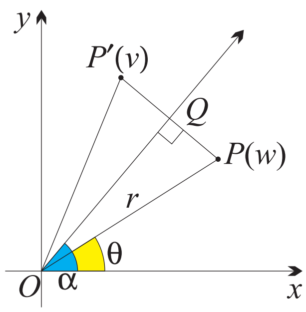
(c) Let $z = x + iy$ and $w = 1 - \dfrac{2i}{z}$.

(i) Write w in the form $a + ib$. **2**

(ii) For what value of z is w undefined? **1**

(iii) Given that w is purely imaginary, describe the locus of z. **2**

(d)



In the Argand diagram above, P represents the complex number $w = r\,\text{cis}\,\theta$. Q is that point on the ray $\arg(z) = \alpha$ such that $\angle PQO = \frac{\pi}{2}$. The point P', which represents the complex number v, is the reflection of P in the ray $\arg(z) = \alpha$. You may assume that $\triangle OPQ \equiv \triangle OP'Q$.

(i) Write down the values of $|v|$ and $\arg(v)$. **2**

(ii) Hence show that $v = \overline{w}\,\text{cis}\,2\alpha$. **1**

(iii) The circle $|z - (2 + 2i)| = 1$ is reflected in the ray $\arg(z) = \frac{\pi}{6}$. By using the result in part (ii), or otherwise, show that the equation of this new circle is **2**

$$\left| z - \left((1 + \sqrt{3}) + i(\sqrt{3} - 1) \right) \right| = 1 .$$

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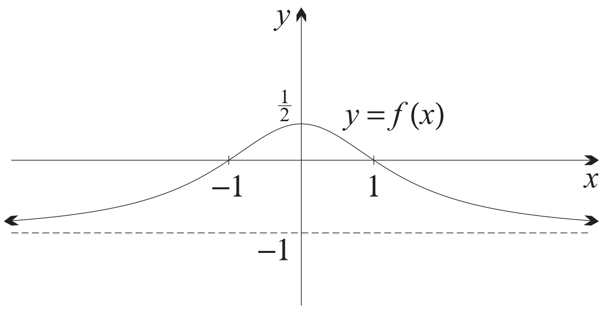
QUESTION THREE (15 marks) Use a separate writing booklet. **Marks**

(a)



The graph of $y = f(x)$ is shown above. The horizontal asymptote is $y = -1$ and the y-intercept is at $(0, \frac{1}{2})$. The x-intercepts are at $(-1, 0)$ and $(1, 0)$.

Draw separate graphs of the following functions:

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = \Big(f(x)\Big)^2$ **2**

(iii) $y = 4^{f(x)}$ **2**

(b) The ellipse $\mathcal{E}$ has equation $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$.

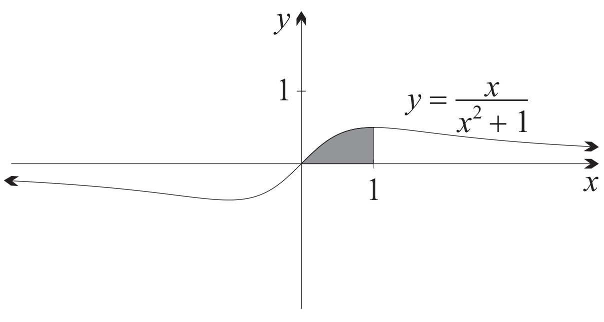
(i) State the intercepts with the axes. **1**

(ii) Determine the eccentricity of $\mathcal{E}$. **1**

(iii) State the coordinates of the two foci. **1**

(iv) Find the equations of the two directrices. **1**

(c) **5**



The graph of $y = \dfrac{x}{x^2 + 1}$ is shown above.

Use the method of cylindrical shells to find the volume of the solid generated when the shaded region bounded by $y = 0$, $y = \dfrac{x}{x^2 + 1}$ and $x = 1$ is rotated about the y-axis.

Exam continues next page ...

QUESTION FOUR (15 marks) Use a separate writing booklet. **Marks**

(a) (i) Show that $2\cos A\,\sin B = \sin(A+B) - \sin(A-B)$. **1**

(ii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{3}} 2\cos 2x\,\sin x\,dx$. **2**

(b) Consider the integral $I_n = \displaystyle\int_0^1 \frac{x^n}{\sqrt{1+x}}\,dx$.

(i) Show that $I_0 = 2\sqrt{2} - 2$. **1**

(ii) Show that $I_{n-1} + I_n = \displaystyle\int_0^1 x^{n-1}\sqrt{1+x}\,dx$. **1**

(iii) Use integration by parts to show that **2**

$$I_n = \frac{2\sqrt{2} - 2nI_{n-1}}{2n+1}.$$

(iv) Hence evaluate I_2. **2**

(c) The number c is real and non-zero. It is also known that $(1+ic)^5$ is real.

(i) Use the binomial theorem to expand $(1+ic)^5$. **1**

(ii) Show that $c^4 - 10c^2 + 5 = 0$. **2**

(iii) Hence show that $c = \sqrt{5-2\sqrt{5}}$, $-\sqrt{5-2\sqrt{5}}$, $\sqrt{5+2\sqrt{5}}$ or $-\sqrt{5+2\sqrt{5}}$. **1**

(iv) Let $1 + ic = r\,\mathrm{cis}\,\theta$. Use de Moivre's theorem to show that the smallest positive value of θ is $\frac{\pi}{5}$. **1**

(v) Hence evaluate $\tan\frac{\pi}{5}$. **1**

Exam continues overleaf ...

QUESTION FIVE (15 marks) Use a separate writing booklet. **Marks**

(a) The polynomial $P(z) = 2z^3 - 3z^2 + 8z + 5$ has a zero at $z = 1 - 2i$. Factorise $P(z)$. **3**

(b) (i) The cubic equation $x^3 - px - q = 0$ has a double root. Show that $27q^2 = 4p^3$. **3**

(ii) Hence find the y-coordinates of the stationary points of $y = x^3 - 3x$ without the use of calculus. **1**

(c) Consider the series:

$$S = 1 - x^2 + x^4 - x^6 + \cdots .$$

(i) For which values of x does S have a limiting sum, and what is the limiting sum? **2**

(ii) Assuming that it is valid to integrate this series, show that **3**

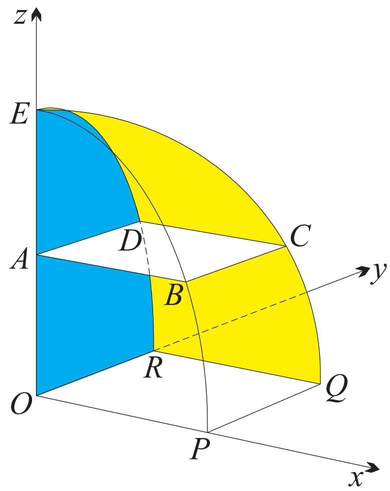
$$\tan^{-1} x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots .$$

(iii) Show that $\tan \frac{\pi}{12} = 2 - \sqrt{3}$. **2**

(iv) Let $x = \tan \frac{\pi}{12}$. Use this value of x and the first three terms of the series in part (ii) to find an approximation for π, correct to four decimal places. **1**

QUESTION SIX (15 marks) Use a separate writing booklet. **Marks**

(a)



The solid in the diagram above has a horizontal square base $OPQR$ with diagonal $OQ = r$. The thin horizontal slice $ABCD$ at height z above the base is also square with $OC = r$. The line OA is vertical. The curve QCE is a quadrant of a circle with centre O and radius r.

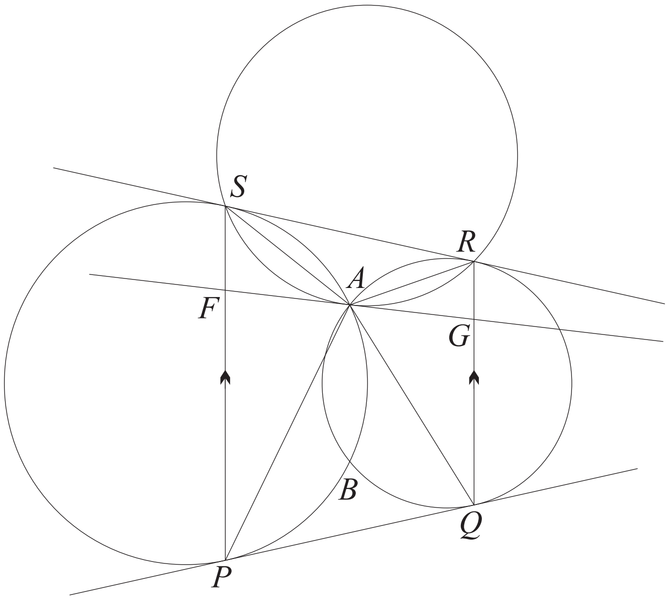
(i) Show that the area of $ABCD$ is $\frac{1}{2}(r^2 - z^2)$. **2**

(ii) Hence find the volume of the solid. **3**

(b) The points $P(a\sec\theta, b\tan\theta)$ and $Q(a\sec\phi, b\tan\phi)$ lie on the same branch of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, and PQ is a focal chord, passing through $S(ae, 0)$. **4**

Use the gradients of PS and QS to show that $e = \dfrac{\sin\theta - \sin\phi}{\sin(\theta - \phi)}$.

(c)



In the diagram above, two circles of differing radius intersect at A and B. The lines PQ and RS are the common tangents with $PS||QR$. A third circle passes through the points S, A and R. The tangent to this circle at A meets the parallel lines at F and G.

Let $\angle RAG = \alpha$, $\angle AGR = \beta$ and $\angle GRA = \gamma$.

NOTE: You do not have to copy the diagram above. It has been reproduced for you on a tear-off sheet at the end of this paper. Insert this sheet into your answer booklet.

(i) State why $\angle AFP = \beta$. **1**

(ii) Show that $\angle SPA = \alpha$. **2**

(iii) Hence prove that FG is also tangent to the circle which passes through the points A, P and Q. **3**

Exam continues next page ...

QUESTION SEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) (i) The definition of ${}^{k}\mathrm{C}_r$ is the coefficient of x^r in the expansion of $(1+x)^k$. Using this definition, what is the value of ${}^{k}\mathrm{C}_r$ whenever $k < r$? **1**

(ii) Prove that $\displaystyle\sum_{k=0}^{n} {}^{k}\mathrm{C}_r = {}^{n+1}\mathrm{C}_{r+1}$. You may assume the addition property for the binomial coefficients, which may be written as ${}^{k}\mathrm{C}_r = {}^{k+1}\mathrm{C}_{r+1} - {}^{k}\mathrm{C}_{r+1}$. **2**

(iii) Use the result proven in part (ii) to show that $\displaystyle\sum_{k=0}^{n} k = \tfrac{1}{2}n(n+1)$. **1**

(iv) (α) Show that $k^2 = 2 \times {}^{k}\mathrm{C}_2 + {}^{k}\mathrm{C}_1$. **1**

(β) Hence find a formula for $\displaystyle\sum_{k=0}^{n} k^2$. **2**

(b) Show that the equation of the directrix of the parabola $y = ax^2 + bx$ is **2**

$$y = -\frac{b^2+1}{4a}.$$

(c) A projectile is fired from the origin O with initial speed V and angle of projection α. The Cartesian equation of its trajectory is

$$y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2V^2}.$$

(i) Use part (b) to find the equation of the directrix. **2**

(ii) Hence show that the focus lies on the circle **1**

$$x^2 + y^2 = \frac{V^4}{4g^2}.$$

(iii) There is only one trajectory which passes through P. Use the geometry of the parabola to prove that OP is a focal chord. **3**

QUESTION EIGHT (15 marks) Use a separate writing booklet. **Marks**

(a) Consider the function

$$f(x) = x - \frac{g^2}{x} - 2g\log\left(\frac{x}{g}\right), \quad \text{for } x \geq g.$$

(i) Evaluate $f(g)$. **1**

(ii) Show that $f'(x) = \left(1 - \frac{g}{x}\right)^2$. **1**

(iii) Explain why $f(x) > 0$ for $x > g$. **1**

(b) A body is moving vertically through a resisting medium, with resistance proportional to its speed. The body is initially fired upwards from the origin with speed v_0. Let y metres be the height of the object above the origin at time t seconds, and let g be the constant acceleration due to gravity. Thus

$$\frac{d^2y}{dt^2} = -(g + kv) \quad \text{where} \quad k > 0.$$

(i) Find v as a function of t, and hence show that **4**

$$k^2 y = (g + kv_0)(1 - e^{-kt}) - gkt.$$

(ii) Find T, the time taken to reach the maximum height. **1**

(iii) Show that when $t = 2T$, **1**

$$k^2 y = (g + kv_0) - \frac{g^2}{g + kv_0} - 2g\log\left(\frac{g + kv_0}{g}\right).$$

(iv) Use this result and part (a) to show that the downwards journey takes longer. **1**

(c) Suppose that the equation $f(x) = 0$ has a single root $x = \alpha$, where $a \leq \alpha \leq b$. Let the sequence

$$x_0 = a,\ x_1 = b,\ x_2 = \frac{a+b}{2},\ x_3,\ x_4,\ \ldots$$

be the successive approximations of $x = \alpha$ obtained when the bisection method is used. (The bisection method is also known as the method of halving the interval.) Let $u_n = |x_n - x_{n-1}|$ be the distances between successive terms of this sequence.

(i) Explain why $u_{n+1} = \frac{1}{2}u_n$. **1**

(ii) Hence show that $u_n = (b - a)\left(\frac{1}{2}\right)^{n-1}$ for $n \geq 1$. **2**

(iii) Explain why $|\alpha - x_n| \leq u_n$. **1**

(iv) Hence prove that the bisection method converges to the root $x = \alpha$. That is, prove that $\lim_{n\to\infty} x_n = \alpha$. **1**

END OF EXAMINATION

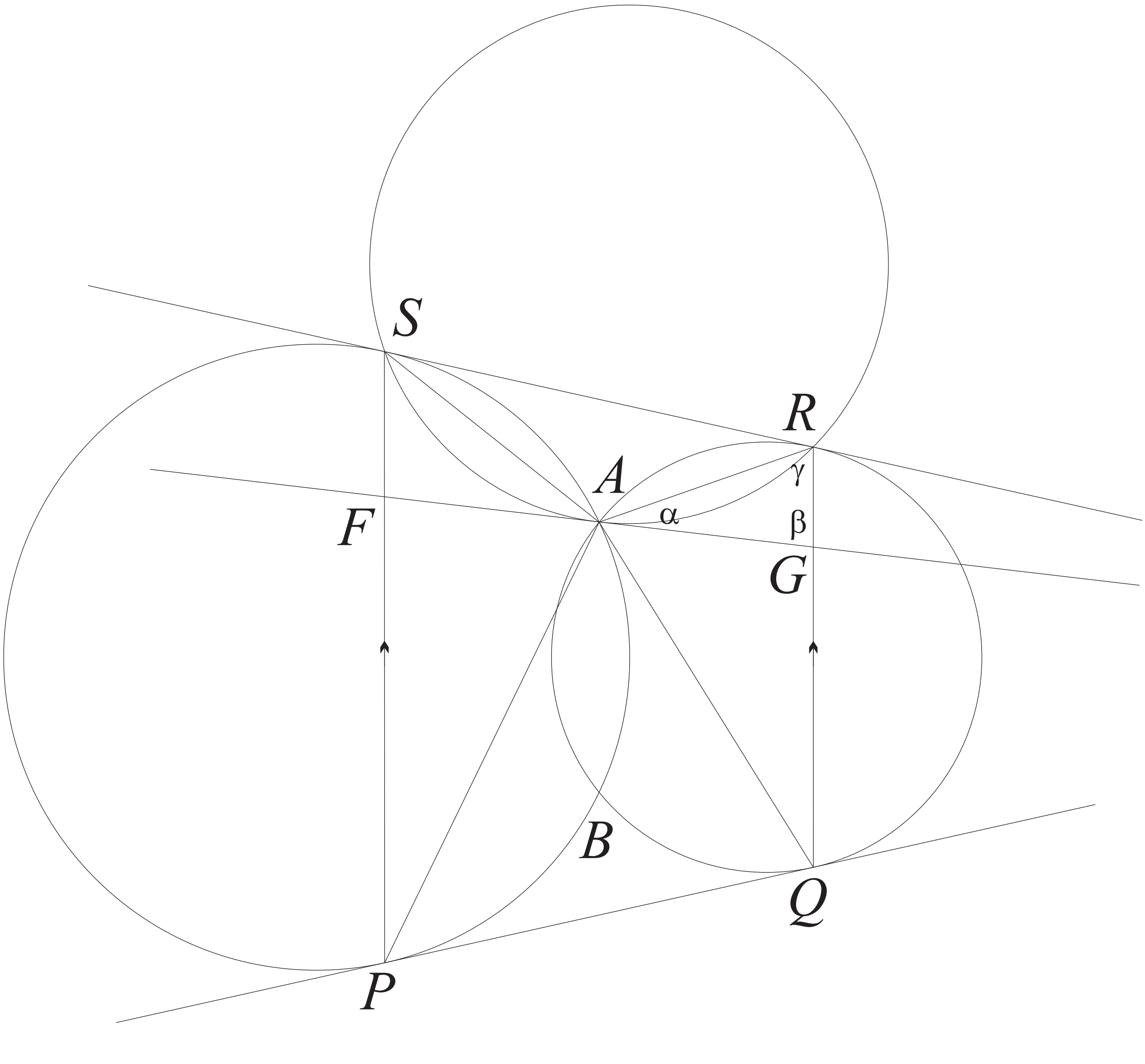
Tear-off pages follow ...

CANDIDATE NUMBER: ....

DETACH THIS SHEET AND BUNDLE IT WITH THE REST OF QUESTION SIX.

QUESTION SIX

(c)



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The following list of standard integrals may be used:

$$\int x^n\,dx = \frac{1}{n+1}\,x^{n+1}, \ \ n \neq -1; \ \ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \ \ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}\,e^{ax}, \ \ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \ \ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \ \ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \ \ a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \ \ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \ \ a > 0, \ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \ \ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

$$\text{NOTE}: \ \ln x = \log_e x, \ x > 0$$