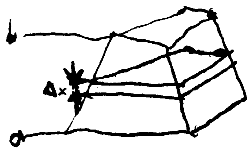
Lecture 45
Volume of Solids

Volume by slicing.



Drawing a cross section should be the same regardless of vertical position when using this method. For small Δx, we have a slice with cross sectional area $A(x)$.

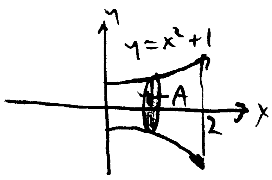
Volume of slice $\approx$ area of base $\times$ thickness

Volume of shape $=$ sum of all the slices.

So volume $\approx \sum_a^b A(x)\ \Delta x$.
So $V = \lim_{\Delta x \to 0} \sum_a^b A(x)\ \Delta x$ taking limits as $\Delta x \to 0$, the volume becomes more accurate and so generally,

$$\boxed{V = \int_a^b A(x)\ dx}$$

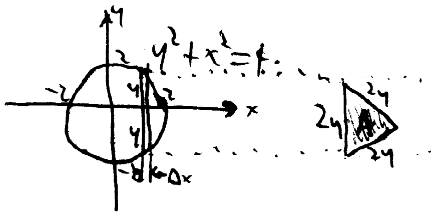
Example 1.



$A = \pi y^2 \therefore V = \int_0^2 \pi y^2\ dx = \int_0^2 \pi(x^2+1)^2\ dx = \frac{206\pi}{15}$.

Example 2. The base of a certain solid is the circle $x^2 + y^2 = 4$. Each plane section of this solid cut out by a plane perpendicular to the x-axis is an equilateral triangle with one side in the base of the solid. Find the volume.

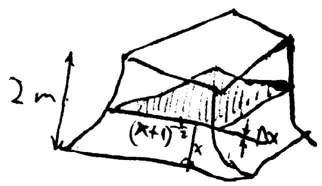
Solution.



Area of $\triangle = \frac{1}{2}ab\sin C = \frac{1}{2}(2y)(2y)\sin 60^\circ = 2y^2\frac{\sqrt{3}}{2} = \sqrt{3}y^2 = \sqrt{3}(4-x^2)$
$\therefore V = \int_{-2}^{2}\sqrt{3}(4-x)^2\,dx = 2\sqrt{3}\int_0^2(4-x^2)\,dx = 2\sqrt{3}[4x-\frac{x^3}{3}]_0^2 = 2\sqrt{3}(4(2)-\frac{2^3}{3}-0) = \frac{32\sqrt{3}}{3}$

Note: When the thickness is Δx, the area of the cross sectional area must be in terms of x. If the thickness is Δy, the area of the cross sectional area must be in terms of y.

Example 3. A figure of height 2 m has cross sections parallel to the base and at a height x metres above the base which are squares of side length given by $S(x) = (x+1)^{-\frac{1}{2}}$. Find the volume of the solid.

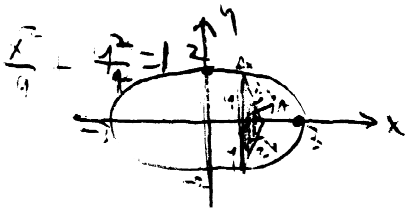


$$\begin{aligned}
\text{Area of slice} &= ((x+1)^{-\frac{1}{2}})^2 \text{ (since it is a square of side length } (x+1)^{-\frac{1}{2}}) \\
&= (x+1)^{-1} \\
&= \tfrac{1}{x+1} \\
\text{So } V &= \int_0^2 \tfrac{dx}{x+1} \\
&= [\ln(x+1)]_0^2 \\
&= \ln 3 \text{ unit}^3
\end{aligned}$$

Lecture 46

Slicing

From Coroneos Supplement Set 3A Q4 - first part



The base of a certain solid S lies in the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$. The cross-section of this solid by planes parallel to the y-axis are equilateral triangles. Find the volume of S.

Solution:

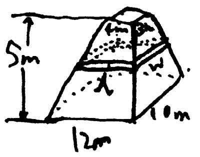
$A = \frac{1}{2}(2y)(2y)\sin 60^\circ = \sqrt{3}y^2,\ y = 4(1 - \frac{x^2}{9}) \Rightarrow$

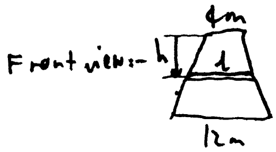
$$
\begin{aligned}
V &= 2\int_0^3 \sqrt{3}y^2\ dx \\
&= 8\sqrt{3}\int_0^3 (1 - \tfrac{x^2}{9})\ dx \\
&= 8\sqrt{3}[x - \tfrac{x^3}{27}]_0^3 \\
&= 8\sqrt{3}(3 - \tfrac{27}{27} - 0) \\
&= 16\sqrt{3}\ \text{unit}^3.
\end{aligned}
$$

Lecture 47

Slices - cont'd

eg. Find the volume of the following:





l is directly proportional to h, so $l \propto h$ $\therefore$ l is a linear function of h.
$\therefore l = mh + b$
When $h = 0,\ l = 4 \Rightarrow b = 4$ and when $h = 5, l = 12 \Rightarrow 12 = 5m + 4$ & $\therefore m = \frac{8}{5}$.
$\therefore l = \frac{8}{5}h + 4$ and similarly $w = \frac{7}{5}h + 3$ and
$V = \int_0^5 lw\ dh$ (since l & w are in terms of h and the slice is a rectangle)

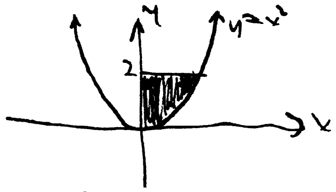
$$= \int_0^5 (\tfrac{8}{5}h + 4)(\tfrac{7}{5}h + 3)\ dh$$
$$= \tfrac{4}{25} \int_0^5 (14h^2 + 65h + 75)\ dh$$
$$= \tfrac{4}{25} [\tfrac{14h^3}{3} + \tfrac{65h^2}{2} + 75h]_0^5$$
$$= \tfrac{4}{25} (\tfrac{14(5)^3}{3} + \tfrac{65(5)^2}{2} + 75(5) - 0)$$
$$= 283 \text{ m}^3 \text{ (to 3 sig. fig.)}.$$

Lecture 48

Volume of solids of revolution.

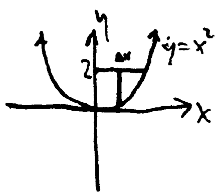
Example 1. Find the volume generated when the area bounded by the curve $y = x^2, y = 2$ and $x = 0$ is rotated about the:-
(a) y-axis **(b)** line $y = 2$

Solution.
(a)



$V = \pi \int_0^2 x^2 \; dy$
$\quad = \pi \int_0^2 y \; dy$
$\quad = [\frac{4}{2} - 0]$
$\quad = 2\pi$ cu. units

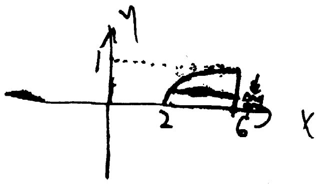
(b)

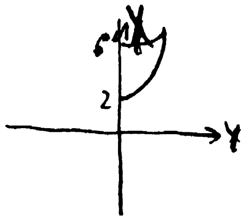


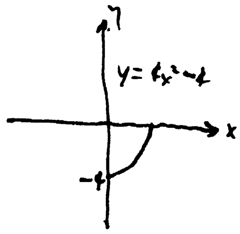
$A(x) = \pi(2 - x^2)^2$
$\therefore V = \pi \int_0^{\sqrt{2}} (2 - x^2)^2 \; dx$
$\quad = \pi \int_0^{\sqrt{2}} (4 - 4x^2 + x^4) \; dx$
$\quad = \pi[4x - \frac{4x^3}{3} + \frac{x^5}{5}]_0^{\sqrt{2}}$
$\quad = \pi(4\sqrt{2} - \frac{4\sqrt{2}^3}{3} + \frac{\sqrt{2}^5}{5})$
$\quad = \pi(4\sqrt{2} - \frac{8}{3}\sqrt{2} + \frac{4}{5}\sqrt{2})$
$\quad = \pi\sqrt{2}(\frac{60-40+12}{15})$
$\quad = \dfrac{32\sqrt{2}\pi}{15}$ cu. units.

Example 2. The region bounded by $y = \frac{1}{2}\sqrt{x-2}$, the x-axis, and the line $x = 6$ is rotated about the line $x = 6$. Show that the volume is $\frac{128\pi}{15}$ cu. units.

Solution.

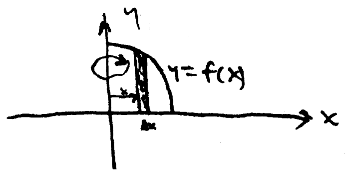




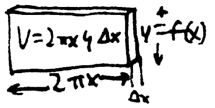


$$
\begin{aligned}
y &= \tfrac{1}{2}\sqrt{x-2} \\
\therefore 2y &= \sqrt{x-2} \\
\therefore x-2 &= 4y^2 \\
\therefore x &= 4y^2+2 \\
\& \therefore A &= \pi(6-(4y^2+2))^2 \\
&= \pi(4-4y^2)^2 \\
&= \pi(4(1-y^2))^2 \\
&= 16\pi(1-y^2)^2 \\
&= 16\pi(1-2y^2+y^4) \\
\therefore V &= \int_0^1 16\pi(1-2y^2+y^4)\ dy \\
&= 16\pi[y-\tfrac{2y^3}{3}+\tfrac{y^5}{5}]_0^1 \\
&= 16\pi[\tfrac{15y-10y^3+y^5}{15}]_0^1 \\
&= 16\pi(\tfrac{15-10+1}{15}) \\
&= 16\pi\tfrac{8}{15} \\
&= \tfrac{128\pi}{15}
\end{aligned}
$$

Lecture 49



If we take a cylinder (around y-axis) and open it, we get:



Note: This method is good because it is revolved around the y-axis, but it is with respect to x.

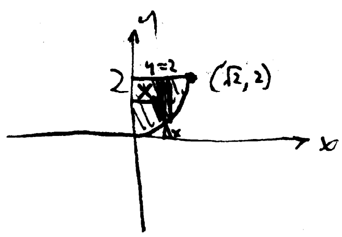
Note: "y" refers to to the height of the cylinder.

Each cylinder has volume $= 2\pi x.y.\Delta x$

Volume of solid $= \sum_0^b 2\pi x.y.\Delta x$

$= \int_0^b 2\pi xy\ dx$

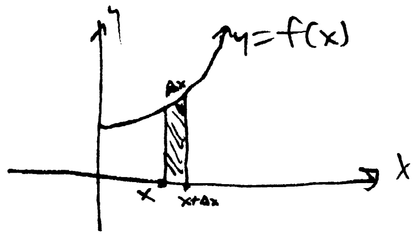
eg. The region bounded by $y = x^2$, the y-axis and the line $y = 2$ is rotated about the y-axis. Find the volume by cylindrical shells.



$$
\begin{aligned}
V &= \int_0^{\sqrt{2}} 2\pi x(2-y)\ dx \\
&= 2\pi \int_0^{\sqrt{2}} x(2-x^2)\ dx \\
&= 2\pi \int_0^{\sqrt{2}} (2x-x^3)\ dx \\
&= 2\pi[x^2 - \tfrac{x^4}{4}]_0^{\sqrt{2}} \\
&= 2\pi(\sqrt{2}^2 - \tfrac{\sqrt{2}^4}{4} - 0) \\
&= 2\pi(2-1) \\
&= 2\pi \text{ cu. units}
\end{aligned}
$$

Lecture 50

Volume by cylindrical shells.

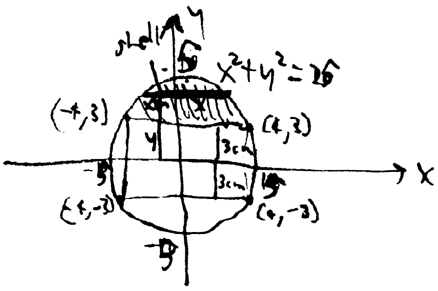


$$\begin{aligned}\Delta V &= \pi(x+\Delta x)^2 y - \pi(x)^2 y\\ &= \pi y(x^2 + 2x\Delta x + (\Delta x)^2 - x^2)\\ &= \pi y(2x\Delta x + (\Delta x)^2)\end{aligned}$$

$\therefore \Delta V = 2\pi xy\ \Delta x$ (($(\Delta x)^2$ is very small for small Δx & $\therefore$ is neglected)

$$\begin{aligned}\therefore V &= \textstyle\sum_a^b 2\pi xy\ \Delta x\\ &= \textstyle\int_a^b 2\pi xy\ dx.\end{aligned}$$

Example. A cylindrical hole of diameter 6cm is drilled through the centre of a sphere of diameter 10cm. Find the volume **(a)** remaining **(b)** cut out.



Solution.
(a)

$\Delta V = 2\pi y(2x)\ \Delta y$

$$\begin{aligned}V &= 4\pi \textstyle\int_3^5 y\sqrt{25-y^2}\ dy \text{ Let } u = 25-y^2 \text{ so } du = -2y\ dy\\ &= -2\pi \textstyle\int_3^5 -2y\sqrt{25-y^2}\ dy\\ &= -2\pi \textstyle\int_{16}^0 u^{\frac{1}{2}}\ du\\ &= -2\pi[\frac{2u^{\frac{3}{2}}}{3}]_{16}^0\\ &= 2\pi[\frac{2}{3}(16)^{\frac{3}{2}} - 0]\\ &= \frac{256\pi}{3} \text{ cu. units}\end{aligned}$$

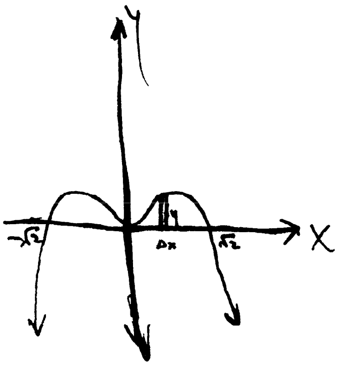
(b) $\therefore$ volume of part cut out is $\frac{4(125)}{3}\pi - \frac{256\pi}{3} = \frac{244\pi}{3}$ cu. units

Lecture 51

Miscellsaneous exercises on volumes

Example 1. Find the volume generated when the region between $y = 2x^2 - x^4$ and the x-axis is rotated about the y-axis, by the bethod of cylindrical shells.

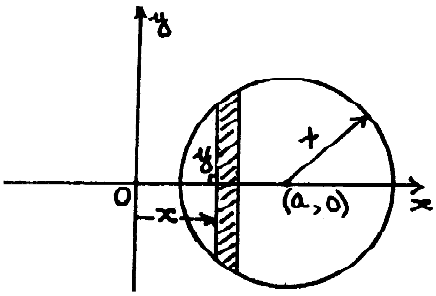
Solution.



$y = x^2(\sqrt{2} - x)(\sqrt{2} + x)$

$$\begin{aligned} V &= \int_0^{\sqrt{2}} 2\pi xy\ dx \\ &= \int_0^{\sqrt{2}} 2\pi x(2x^2 - x^4)\ dx \\ &= 2\pi \int_0^{\sqrt{2}} (2x^3 - x^5)\ dx \\ &= 2\pi[\tfrac{2x^4}{4} - \tfrac{x^6}{6}]_0^{\sqrt{2}} \\ &= 2\pi(\tfrac{2\sqrt{2}^4}{4} - \tfrac{\sqrt{2}^6}{6}) \\ &= \tfrac{4\pi}{3}\ \text{unit}^3 \end{aligned}$$

Example 2. From Coroneos Supplement Set 3D Q17i



The circle $(x - a)^2 + y^2 = r^2,\ (a > r)$ is rotated about the y-axis to form an anchor-ring or torus. By considering the rotation of the strip of area of thickness δx shown about the y-axis, prove the volume V of the anchor-ring is given by $V = 4\pi \int_{a-r}^{a+r} x.\sqrt{r^2 - (x - a)^2}.dx$ and hence find V. {Hint: Let $x - a = r\sin\theta$}

Solution.

Where $y = \sqrt{r^2 - (x-a)^2}$,

$$\begin{aligned}
V &= 2\int_{a-r}^{a+r} 2\pi xy\ dx \\
&= 4\pi \int_{a-r}^{a+r} x\sqrt{r^2-(x-a)^2}\ dx \\
&= 4\pi \int_{-\pi/2}^{\pi/2} (r\sin\theta + a)\sqrt{r^2 - r^2\sin^2\theta}\, r\cos\theta\ d\theta \text{ (where } x-a = r\sin\theta) \\
&= 4\pi \int_{-\pi/2}^{\pi/2} (r\sin\theta + a) r\cos\theta\, r\cos\theta\ d\theta \\
&= 4\pi r^2 \int_{-\pi/2}^{\pi/2} (r\sin\theta\cos^2\theta + a\cos^2\theta)\ d\theta \\
&= 4\pi r^2 (0 + 2a\int_0^{\pi/2} \tfrac{1}{2}(\cos 2\theta + 1)\ d\theta) \text{ (}\sin\theta\cos^2\theta \text{ is odd, } \cos^2\theta \text{ is even)} \\
&= 4\pi a r^2 [\tfrac{1}{2}\sin 2\theta + \theta]_0^{\pi/2} \\
&= 4\pi r^2 (\tfrac{a\pi}{2}) \\
&= 2\pi^2 a r^2 \text{ unit}^3.
\end{aligned}$$