Question 1 Western Region 1993 4Unit trial

a. Find:

i. $\displaystyle\int \frac{1-2x}{\sqrt{1-x^2}} \, dx$

ii. $\displaystyle\int x\sqrt{x-3} \, dx$

b. Evaluate:

i. $\displaystyle\int_0^{\frac{\pi}{4}} \theta \cos^2\theta \, d\theta$

ii. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{5+4\cos x}$

c. i. Write $\dfrac{40}{(x+1)(x^2+4)}$ in the form $\dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+4}$

ii. Hence show that

$$\int_0^2 \frac{40\,dx}{(x+1)(x^2+4)} = \pi - 4\ln 2 + 8\ln 3$$

Question 2

a. If $y=0$ when $x=0$ and $(1-x^2)\dfrac{dy}{dx} = 2$

show that $x = \dfrac{e-1}{e+1}$ when $y=1$

b. If $I_n = \int_0^1 x^n e^x \, dx$, (n positive integer)

show that $I_{n+1} = e - (n+1) I_n$

c. i. Write expressions for sin (x + y) and sin (x − y) in terms of sin x, sin y, cos x and cos y.

ii. Show that $\sin(ax + x) - \sin(ax - x) = 2 \cos ax \sin x$

iii. Prove that

$$\cos x + \cos 3x + \cos 5x + \ldots \cos(2n-1)x = \frac{\sin 2n x}{2 \sin x}$$

Question 3

a. On separate diagrams draw a neat sketch of the locus specified by each of the following. Include a description to clarify your sketch where necessary.

i. $0 \le \text{Arg}(z+1) \le \dfrac{\pi}{4}$

ii. $\left| \dfrac{z+1}{z-1} \right| = 1$

iii. $\text{Arg}\left(\dfrac{z+1}{z-1}\right) = \dfrac{\pi}{4}$

b. i. Show that $\dfrac{\pi}{6} + \dfrac{\pi}{4} = \dfrac{5\pi}{12}$ and hence that $\tan \dfrac{5\pi}{12} = \dfrac{\sqrt{3}+1}{\sqrt{3}-1}$

Find a similar expression for $\tan \dfrac{\pi}{12}$

ii. Express $\sqrt{-6i}$ in the form $a + ib$ where a and b are real.

iii. Solve $z^2 + (1+i)z + 2i = 0$ expressing the roots in the form $x + iy$ where x and y are real. If these roots are z_1 and z_2 prove that

$$|z_1| = |z_2| = \sqrt{2} \text{ and } \arg z_1 + \arg z_2 = \frac{\pi}{2}$$

Question 4

The ellipse E has equation $4x^2 + 9y^2 = 36$

i. Sketch the ellipse E indicating its foci S, S' and its directrices.

ii. Show that the point $P(3\cos\theta, 2\sin\theta)$ lies on E.

iii. Derive the equation of the tangent to the ellipse at P and find the co-ordinates of Q, the point where this tangent cuts the major axis.

iv. The normal to the ellipse E at point P cuts the major axis at R. Find the co-ordinates of R.

v. The line through P, parallel to the y axis meets the major axis at T. O is the centre of the ellipse. Show that $OQ.RT$ is a constant.

Question 5

a. Consider the function $f(x) = \frac{1}{2}(e^x + e^{-x})$

i. Find any stationary points and determine their nature.

ii. Describe the behaviour of $f(x)$ as $x \to \pm\infty$.

iii. Make a neat sketch of $y = f(x)$ (<u>not</u> on graph paper).

iv. Given that the length of the curve of $y = f(x)$ from $x = a$ to $x = b$ is given by

$$\text{Length} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

find the length of the curve $f(x) = \frac{1}{2}(e^x + e^{-x})$ from

$x = 0$ to $x = \ln 2$.

b. Through the vertices A, B, C of a given triangle are drawn lines B' C', C' A' and A' B' respectively to form an equilateral triangle A' B' C' which circumscribes the triangle ABC.

i. Draw a diagram to represent this information.

ii. If $A\hat{C}B' = \theta$ and $A\hat{B}C' = \alpha$

show that $B'C' = \frac{2}{\sqrt{3}}(b\sin\theta + c\sin\alpha)$

Question 6

a. The polynomial $P(x) = x^3 + ax^2 + bx + 6$, (a,b real), has $(1 - i)$ as one zero. Find a and b and hence factorise $P(x)$ over the complex field C.

b. If α, β, γ are the roots of $x^3 + 2x^2 - 2x - 4 = 0$

find $$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$$

c. i. Write an expression to find the volume when a region bounded by the curve $y = f(x)$ and the x - axis, between $x = a$ and $x = b$ is rotated about the y axis, using the Shell method.

ii. Using the method of cylindrical shells, find the volume formed when the region (in the first quadrant) between the curves $y = 6x - 3x^2$ and $y = 3x$ is rotated about the y - axis.

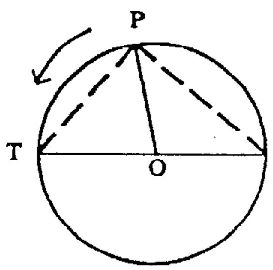
Question 7

a. A light inextensible string of length 7a units has its end attached to two fixed points A and B with A distant 5a units vertically above B.
A bead C of mass m is threaded onto the string and it rotates in a horizontal circle with constant angular velocity ω. AC = 4a.
The acceleration due to gravity is g.

i. Show that $\omega^2 = \dfrac{35g}{12a}$

ii. Find the time taken by C to complete one revolution.

b.



P is moving in a circle centre O with angular velocity of $\frac{d\beta}{dt}$ about the centre

Its angular velocity about a fixed point Q, is $\frac{d\alpha}{dt}$

i. Show that $\frac{d}{dt}(2\alpha) = \frac{d}{dt}(180^\circ + \beta)$

What conclusion do you make regarding the angular velocities about O and Q?

Question 8

a. i In how many ways can 10 students be grouped into two teams of 5 to play a game of basketball?

ii. Two of the 10 students are twins. If the teams are formed at random, what is the probability that the twins play on the same team?

b. i. Write expressions for $\cos 2\theta$ and $\sin 2\theta$ in terms of $\sin\theta$ and $\cos\theta$ and hence find $\cos\dfrac{2\pi}{n}$ and $\sin\dfrac{2\pi}{n}$ in terms of $\sin\dfrac{\pi}{n}$ and $\cos\dfrac{\pi}{n}$

ii. Using the results of (i) and De Moivre's Theorem

prove that $\left(1+\cos\dfrac{2\pi}{n}+i\sin\dfrac{2\pi}{n}\right)^n = -2^n\cos^n\dfrac{\pi}{n}$

c. Three positive numbers a, b and c satisfy the conditions that $a \geq b \geq c$ and $a+b+c \leq 1$.
By considering $(a+b+c)^2$, or otherwise, prove that $a^2+3b^2+5c^2 \leq 1$