1995 Western Region Trial 4U

QUESTION 1

MARKS

a) Find

i) $\int \frac{dx}{x^2 - 4x + 20}$ (2)

ii) $\int \sin^3 x \, dx$ (2)

b) Evaluate $\int_0^1 \frac{x \, dx}{\sqrt{4-x}}$ (3)

c) Show that $\int_e^{e^2} 2x \log_e x \, dx = \frac{e^2}{2}\left(3e^2 - 1\right)$ (3)

d) i) Find real numbers a, b and c such that (3)

$$\frac{2x+3}{(x^2+1)(x+2)} = \frac{ax+b}{x^2+1} + \frac{c}{x+2}$$

ii) Hence find $\int \frac{2x+3}{(x^2+1)(x+2)} \, dx$ (2)

QUESTION 2

MARKS

a) Express in modulus - argument form

i) $\sqrt{3}-i$ (2)

ii) $(\sqrt{3}-i)^4$ (1)

b) Find all real pairs of integers x and y such that

$$(x+iy)^2 = 5-12i$$ (2)

Hence or otherwise solve the quadratic equation

$$z^2+4z-1+12i=0$$ (2)

c) Sketch the locus of those points z such that

i) $0 \le \arg z < \frac{\pi}{2}$ *and* $|z| > 2$ (2)

ii) $\operatorname{Im}(z^2) \ge 1$ (2)

d) What is the maximum value of $|z|$ for $|z-1-i| \le 2$ (2)

e) If $|z_1 - z_2| = |z_1 + z_2|$ show that $\arg z_1 - \arg z_2 = \frac{\pi}{2}$ (2)

QUESTION 3

MARKS

a) The point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. The normal at P cuts the x-axis at G and N is the foot of the perpendicular from P to the x-axis.

i) Prove that the equation of the normal at P is $ax\sin\theta+by=(a^2+b^2)\tan\theta$. (3)

ii) Show that $OG = e^2\ 0N$ where 0 is the origin and e is the eccentricity of the hyperbola. (3)

iii) Show also that $SG = e \times SP$ where S is the focus of the hyperbola. (3)

b) In a circle, centre 0, a diameter AB and a chord AC are drawn. D is a point on the circumference on the side of AB opposite to C, such that the tangent at D is perpendicular to AC.

i) Draw a sketch to show all this information. (1)

Prove that

ii) DC = DB (2)

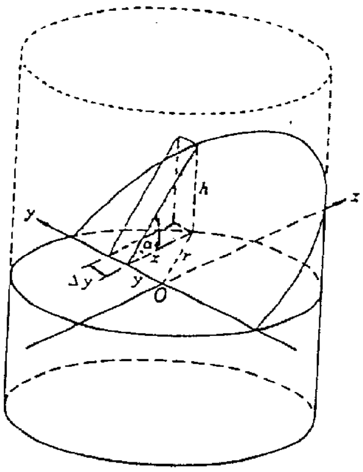
iii) DO produced bisects BC (1)

iv) $\angle ADC + 2\angle ACD = \frac{\pi}{2}$ (2)

QUESTION 4

MARKS

a) A wedge is cut from a right circular cylinder of radius r by two planes, one perpendicular to the axis of the cylinder while the second makes an angle α with the first and intersects it at the centre of the cylinder.



A is the area of the triangle that forms one face of the slice.

i) Show that $A = \frac{1}{2}\tan\alpha\left(r^2 - y^2\right)$ (4)

ii) Hence show that the volume of the wedge is

$$\frac{2}{3}r^3\tan\alpha$$ (3)

b) The polynomial $Q(x) = x^3 - 6x^2 + ax + b$ has one root $1 - i\sqrt{5}$, where a and b are real.

i) Find the values of a and b (4)

ii) Hence solve the equation $Q(x) = 0$ (1)

c) Form a new equation whose roots are one more than those of the equation $x^3 - 2x - 3 = 0$ (3)

QUESTION 5

MARKS

a) Find the limiting sum of the series $\frac{1}{5}+\frac{2}{5^2}+\frac{3}{5^3}+\ \ldots\ldots$ (3)

b) Given $S_R=\sum_{K=1}^{R}K^2$ prove by mathematical induction or otherwise that

$$N\,S_N-\sum_{R=1}^{N-1}S_R=\sum_{R=1}^{N}R^3 \quad \text{for integral } N>1.$$

(5)

c) Find all x such that $\sin x=\cos 4x \qquad 0<x<\pi$ (3)

d) A car is travelling round a section of a race track which is banked at an angle of 15°. The radius of the track is 100 metres. What is the speed at which the car can travel without tending to slip? (4)

QUESTION 6 | MARKS

A) Let $U_n = \int_0^{\frac{\pi}{2}} \cos^n \theta \, d\theta$

i) Prove $U_n = \frac{n-1}{n} \cdot U_{n-2} \text{ for } n \geq 2$ (3)

ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \cos^9 \theta \, d\theta$ (2)

b) For the function $f(x) = \frac{(x-2)}{(x+1)(x-1)}$

i) Sketch $f(x)$ showing the location and nature of all stationary points and the equations of all asymptotes. (Let $\sqrt{3} = 1 \cdot 7$) (5)

ii) On the same axis sketch $g(x) = |f(x)|$ (1)

iii) Using the graph or otherwise solve $f(x) > 0$ for all x. (1)

c) A particle moves under gravity for which the resistance to its motion is directly proportional to the product of its mass and velocity. (v)

i) Show that $\frac{dv}{dt} = g - vk$ where k is a constant. (1)

ii) If the particle falls vertically from rest, find its terminal velocity. (2)

QUESTION 7

MARKS

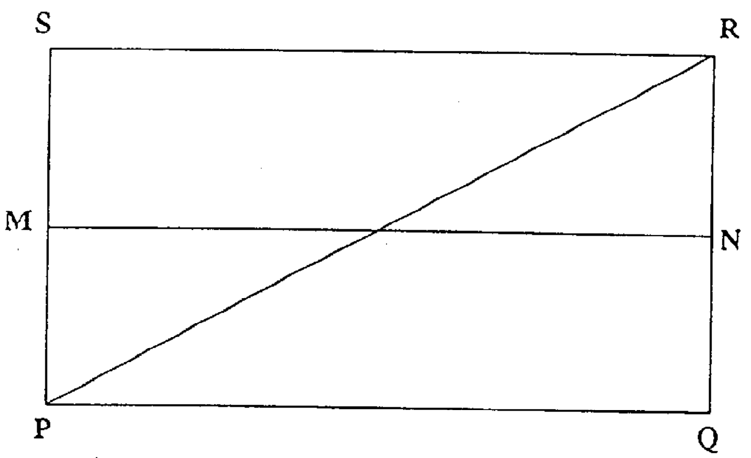
a) Solve for y: $2^{3y+1} = 5^{y+1}$ to 3 significant figures. (2)

b) i) Obtain the equation of the tangent to curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$ at the point P(a,b) on the curve. (2)

ii) This tangent meets the x and y axes at Q and R respectively. Show that OQ + OR = c for all positions of P, where O is the origin. (3)

c) Solve the inequality $\dfrac{x+3}{x-1} \geq x$ (2)

d) The Minister for Education wants to instal a rectangular notice board PQRS of fixed area A square metres in each of her offices. The notice board is to be subdivided by two thin strips of red tape PR and MN (where MN is parallel to PQ) as shown below.



i) Show that the length L of the red tape can be expressed as $L = x + \sqrt{x^2 + \dfrac{A^2}{x^2}}$ where x is the length of PQ. (2)

ii) Find in terms of A, the dimensions of the notice board so that the length of the red tape is a minimum. (4)

QUESTION 8

MARKS

a) A city council consists of 6 liberal and 5 labour aldermen, from which a committee of 5 members is chosen at random. What is the probability that the liberals have a majority on the committee. (2)

b) i) Factorise $1+x+x^2+x^3$ (1)

ii) For the function $P(x)=\frac{x^4}{4}+\frac{x^3}{3}+\frac{x^2}{2}+x+c$

Prove that P(x) has no real roots if $C>\frac{7}{12}$

How many real roots are there if $C\le\frac{7}{12}$ (5)

c) i) Show that $\frac{t^4(1-t)^4}{1+t^2}=t^6-4t^5+5t^4-4t^2+4-\frac{4}{1+t^2}$ (2)

ii) Hence, evaluate $\int_0^1\frac{t^4(1-t)^4\,dt}{1+t^2}$ (2)

iii) Using the integrals $\int_0^1\frac{t^4(1-t)^4\,dt}{1+t^2}$ and $\int_0^1 t^4(1-t)^4\,dt$

establish the inequality (3)

$$\frac{1}{1260}<\frac{22}{7}-\pi<\frac{1}{630}$$