Question 1 (15 marks) Commence each question on a SEPARATE page

a. Find $\int \frac{dx}{x \log_e x}$ **2**

b. Find $\int \frac{dx}{\sqrt{3+2x-x^2}}$ **2**

c. Find $\int \frac{dx}{(x+1)(x^2+4)}$ **4**

d. Using the substitution $t = \tan \frac{x}{2}$, calculate $\int \frac{15}{17+8\cos x}\, dx$, leaving your answer in terms of t. **3**

e. i. Differentiate $\frac{x}{\sqrt{x-3}}$. **1**

ii. Hence evaluate $\int_4^7 \frac{2x-9}{2(x-3)\sqrt{x-3}}\, dx$ **3**

Question 2 (15 marks) Commence each question on a SEPARATE page

a. i. Express $-1 + i\sqrt{3}$ in modulus argument form. **2**

ii. Hence evaluate $(-1 + i\sqrt{3})^{-6}$ **2**

b. If z is a non-zero complex number such that $z + \frac{1}{z}$ is real, prove that $Im(z) = 0$ or $|z| = 1$. **3**

c. Sketch the region where the inequalities $-\frac{\pi}{2} \leq \arg(z - 1 - 2i) \leq \frac{\pi}{4}$, and $|z| \leq \sqrt{5}$ both hold. **3**

d. Let z be a complex number for which $|z| = 1$ and $\arg z = \theta$, where $0 < \theta < \frac{\pi}{2}$.

i. Show that $|1 - z| = \sqrt{2 - 2\cos\theta}$ and $|1 + z| = \sqrt{2 + 2\cos\theta}$ **3**

ii. Hence find the value of $\left|\frac{2}{1 - z^2}\right|$ in terms of θ. **2**

Question 3 (15 marks) Commence each question on a SEPARATE page

a. Show that $\int_0^{\frac{\pi}{4}} x \sin x \, dx = \frac{\sqrt{2}}{8}(4 - \pi)$ **3**

b. The shape of a particular cake can be represented by rotating the region between the curve $y = \sin x$ and the x-axis, from $x = 0$ and $x = \frac{\pi}{4}$, about the line $x = \frac{\pi}{4}$. Using the method of cylindrical shells, find the volume of the cake. **4**

c. The hyperbola H has equation $9x^2 - 4y^2 = 36$.

i. Find the co-ordinates of the foci, S and S′. **2**

ii. Find the equations of the directrices. **1**

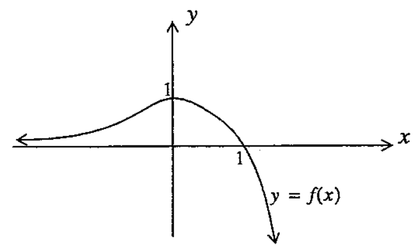
iii. Find the equations of the asymptotes. **1**

iv. Sketch the curve H indicating the information obtained in i. to iii. **1**

v. The point $P(x_0, y_0)$ lies on H. Prove that the equation of the tangent at P is $9x_0x - 4y_0y = 36$. **3**

Question 4 (15 marks) Commence each question on a SEPARATE page

a. The graph of $y = f(x)$ is sketched below.
There is a stationary point at (0, 1)



Use this graph to sketch the following without calculus, showing essential features.

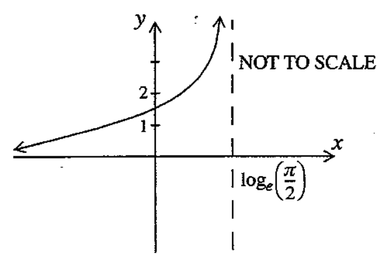
i. $y = f\left(\frac{x}{2}\right)$ **1**

ii. $y = x + f(x)$ **2**

iii. $y = \frac{1}{f(x)}$ **2**

iv. $y = f\left(\frac{1}{x}\right)$ **2**

b. The diagram shows part of the curve $y = \tan(e^x)$, where $x < \log_e\left(\frac{\pi}{2}\right)$. The part to the right of $\log_e\left(\frac{\pi}{2}\right)$ has not yet been drawn.



i. By considering values of x greater than $\log_e\left(\frac{\pi}{2}\right)$, find the smallest positive solution to the equation $\tan(e^x) = 0$. **1**

ii. Copy the diagram and hence sketch the curve $y = \tan(e^x)$ for $x < \log_e\left(\frac{3\pi}{2}\right)$. **1**

iii. How many solutions are there to the equation $\tan(e^x) = 0$ in the domain $1 < x < 3$? **2**

iv. Find the equation of the inverse function of the $y = \tan(e^x)$ for the case when

α. $x < \log_e\left(\frac{\pi}{2}\right)$. **2**

β. $\log_e\left(\frac{\pi}{2}\right) < x < \log_e\left(\frac{3\pi}{2}\right)$ **2**

Question 5 (15 marks) Commence each question on a SEPARATE page

a. i. Prove that $\tan^{-1} n - \tan^{-1}(n - 1) = \tan^{-1}\dfrac{1}{n^2 - n + 1}$, where n is a positive integer. **2**

ii. Hence evaluate $\tan^{-1} 1 + \tan^{-1}\dfrac{1}{3} + \ldots + \tan^{-1}\dfrac{1}{n^2 - n + 1}$. **2**

iii. Hence find the limit $\displaystyle\sum_{n=1}^{\infty} \tan^{-1}\frac{1}{n^2 - n + 1}$. **1**

b. A food package of mass m kg has a parachute device attached. It is released from rest from the top of a cliff 100 metres high. During its fall, the only forces acting are gravity, and owing to the parachute, a resistive force of magnitude $\dfrac{1}{10}mv^2$, where v metres per second is the speed of the package

After $\dfrac{1}{2}\ln 99$ seconds, the parachute disintegrates, and then the only force acting on the particle is due to gravity.

The acceleration due to gravity is taken as 10 m s^{-1}. At time t seconds after being dropped, the package has fallen a distance of x metres from the plane, and its speed is $v\,\text{m s}^{-1}$.

i. Show that while the parachute is operating, $\ddot{x} = 10 - \dfrac{v^2}{10}$. Hence show that $v = 10\left(\dfrac{e^{2t} - 1}{e^{2t} + 1}\right)$ and $x = 5\ln\left(\dfrac{100}{100 - v^2}\right)$ **5**

ii. Find the exact speed of the package and the exact vertical distance fallen just before the parachute disintegrates. **2**

iii. Find the speed of the package just before it reaches the ground. Give your answer correct to two significant figures. **3**

Question 6 (15 marks) Commence each question on a SEPARATE page

a. The polynomial $P(z)$ has equation $P(z) = z^4 - 2z^3 - z^2 + 2z + 10$ **3**
Given that $z - 2 + i$ is a factor of $P(z)$, express $P(z)$ as a product of two quadratic factors with real coefficients.

b. A particle moves in a straight line. It is placed at the origin on the x-axis and is then released from rest. When at position x, its acceleration is given by

$$\ddot{x} = -9x + \frac{5}{(2-x)^2}.$$

i. Show that $v^2 = \dfrac{x(3x-5)(3x-1)}{2-x}$. **2**

ii. Prove that the particle moves between two points on the x-axis, and find these points. **4**

c. An athlete is throwing a javelin. The horizontal and vertical components of the speed of the javelin after t seconds are:

$$\dot{x} = V + 3V\cos\theta \text{ and } \dot{y} = 3V\sin\theta - gt$$

where V is a positive constant, θ is an acute angle, and x and y are the horizontal and vertical displacements from the point of projection.
(Assume when $t = 0$, $x = 0$ and $y = 0$)
Show that:

i. $x = Vt + 3Vt\cos\theta$ and $y = 3Vt\sin\theta - \frac{1}{2}gt^2$. **2**

ii. the range of the javelin, R metres, is given by $R = \dfrac{6V^2\sin\theta}{g}(3\cos\theta + 1)$. **2**

iii. the angle θ which will yield maximum range is $\theta = \cos^{-1}\left(\dfrac{\sqrt{73}-1}{12}\right)$. **3**

Question 7 (15 marks) Commence each question on a SEPARATE page

a. Given that the quartic polynomial $p(x) = x^4 - 5x^3 - 9x^2 + 81x - 108$ has a zero of multiplicity three, factorise the polynomial completely and find all its zeroes. **3**

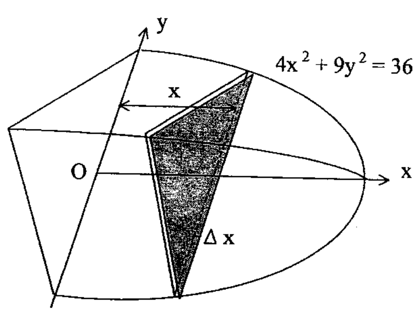
b. Let $Q(x) = x^3 + px + q$, where p and q are real and non-zero. Two of the zeroes of $Q(x)$ are $a + ib$ and k, where a, b and k are real and non-zero and $k < 0$.
It is known that the graph of $y = Q(x)$ has two turning points.

i. By a consideration of $Q'(x)$, show that $p < 0$. **1**

ii. Deduce that $a > 0$. **2**

iii. Show that $q = 8a^3 + 2ap$ **3**

c.



The base of the solid **K** shown in the diagram is the region in the x-y plane enclosed between the semi-ellipse $4x^2 + 9y^2 = 36$ and the y-axis. Each cross section perpendicular to the x-axis is an equilateral triangle.

i. Consider a slice of the solid with thickness Δx and distance x from the y-axis. Find the area of this slice in terms of x. **2**

ii. Find the volume of the solid **K**. **2**

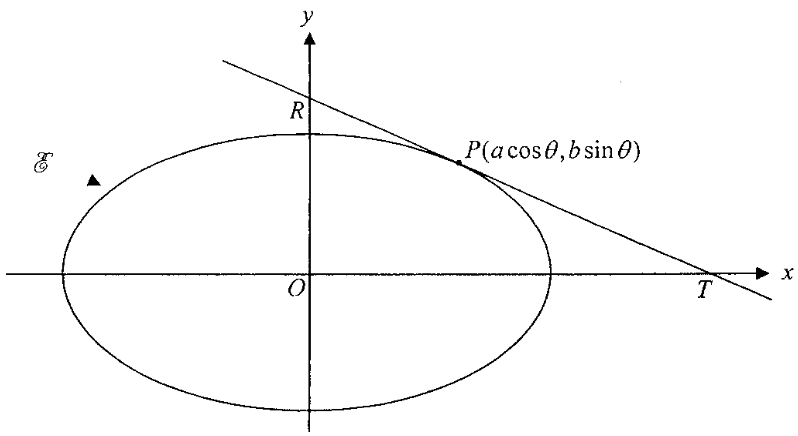
iii. Solid **J** has the same base as solid **K** but its perpendicular cross sectional slice is an isosceles right angled triangle with its hypotenuse in the x-y plane. Find the ratio of volumes of solid **K** to solid **J**. **2**

Question 8 (15 marks) Commence each question on a SEPARATE page

a. A particle P of mass m moves with constant angular velocity ω on a circle of radius r. Its position at time t is given by: $x = r\cos\theta \quad y = r\sin\theta$, where $\theta = \omega t$. **3**

Show that there is an inward radial force of magnitude $mr\omega^2$ acting on P.

b.



The ellipse ℰ with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ shown in the diagram above, has a tangent at the point P($a\cos\theta$, $b\sin\theta$). The tangent cuts the x-axis at T and the y-axis at R.

i. Show that the equation of the tangent at the point P is **2**

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1.$$

ii. If T is the point of intersection between the tangent at point P and one of the directrices of the ellipse, show that $\cos\theta = e$. **2**

iii. Hence find the angle that the focal chord through P makes with the x-axis. **1**

iv. Using similar triangles or otherwise, show that $RP = e^2RT$. **3**

c. Let $I_n = \int_0^1 x(x^2 - 1)^n \, dx$ for $n = 0, 1, 2, \ldots$

Use integration by parts to show that $I_n = \frac{-n}{n+1} I_{n-1}$ for $n \geq 1$. **4**