2020 Mathematics Extension 2 HSC Q16b Alternative Solution

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This question can be done in reverse order by use of properties of beta functions and without any integration whatsoever.

iv. The beta function $B(z_1, z_2) := \int_0^1 t^{z_1-1}(1-t)^{z_2-1}\,dt$ where $\Re(z_1) > 0$ and $\Re(z_2) > 0$. Then where Γ is the gamma function we have $B(z_1, z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}$ and also for non-negative integers x, $\Gamma(x+1) = x!$ whereupon $B(n+1, n+1) = \frac{(n!)^2}{(2n+1)!}$

The statement in the question is equivalent to proving $B(n+1, n+1) \leq 4^{-n}$

$B(1,1) = \frac{(0!)^2}{(2\times 0+1)!} = 1 \leq 4^{-0}$ $\therefore$ it is true for $n = 0$

If it is true for $n = k$ then $B(k+1, k+1) \leq 4^{-k}$

$$\begin{aligned}
\therefore B(k+2, k+2) &= \frac{((k+1)!)^2}{(2k+3)!} \\
&= \frac{(k+1)^2}{(2k+3)(2k+2)} \cdot \frac{(k!)^2}{(2k+1)!} \\
&\leq \frac{(k+1)^2}{(2k+2)(2k+2)} \cdot B(k+1, k+1) \\
&\leq \frac{1}{2} \cdot \frac{1}{2} \cdot 4^{-k} \\
&= 4^{-(k+1)} \text{ and then it is true for } n = k+1
\end{aligned}$$

So by the principle of mathematical induction, $B(n+1, n+1) \leq 4^{-n}$ for all non-negative integers n and so $(2^n n!)^2 \leq (2n+1)!$

iii. $J_n = B(n+1, n+1) = \frac{(n!)^2}{(2n+1)!}$

ii. We also have $B(z_1, z_2) = 2\int_0^{\frac{\pi}{2}} \sin^{2z_1-1}\theta \cos^{2z_2-1}\theta\, d\theta$ whereupon $I_n = \int_0^{\frac{\pi}{2}} 2^{2n+1} \sin^{2n+1}\theta \cos^{2n+1}\theta\, d\theta = 2^{2n} B(n+1, n+1) = \frac{2^{2n}(n!)^2}{(2n+1)!}$.

i.
$$\begin{aligned}
I_n &= \frac{2^{2n}(n!)^2}{(2n+1)!} \\
&= \frac{2^{2n}(n(n-1)!)^2}{(2n+1)(2n)(2n-1)!} \\
&= \frac{2n}{2n+1} \cdot \frac{2^{2n-2}((n-1)!)^2}{(2n-1)!} \\
&= \frac{2n}{2n+1} I_{n-1} \text{ for } n \geq 1
\end{aligned}$$