NORTH SYDNEY BOYS HIGH SCHOOL

2023 YEAR 12 HSC ASSESSMENT TASK 3

Mathematics Advanced

General Instructions

- Reading time – 10 minutes
- Working time – **3 hours**
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Attempt all questions

Class Teacher:
(Please tick or highlight)

O Ms Sarofim
O Ms Fu
O Mr Lin
O Mr Ireland
O Dr Vranesevic

Student Number:

(To be used by the exam markers only.)

Question No	1-10	11-30	Total
Mark	/10	/90	/100

Section I

10 marks

Allow about 15 minutes for this section. Use the multiple-choice answer sheet

1 The period of the function $f(x) = 2\tan(4x - \frac{\pi}{3})$ is

A $\frac{\pi}{2}$

B $\frac{\pi}{3}$

C $\frac{\pi}{4}$

D π

2 For what values of x is the curve $f(x) = x^4 - 2x^3$ concave down?

A $[0, \frac{3}{2}]$

B $\left(0, \frac{3}{2}\right)$

C $(0, 1)$

D $[0, 1]$

3 Which expression is the derivative of $cos^2 3x$ when differentiated with respect to x?

A $-6\sin 3x \cos 3x$

B $-2\sin 3x \cos 3x$

C $2\sin 3x \cos 3x$

D $6\sin 3x \cos 3x$

4 The amount M of a drug present in the blood after t hours is given by

$M = 9t^2 - t^3$ for $0 \leq t \leq 9$.

When is the amount of drug in the blood increasing most rapidly?

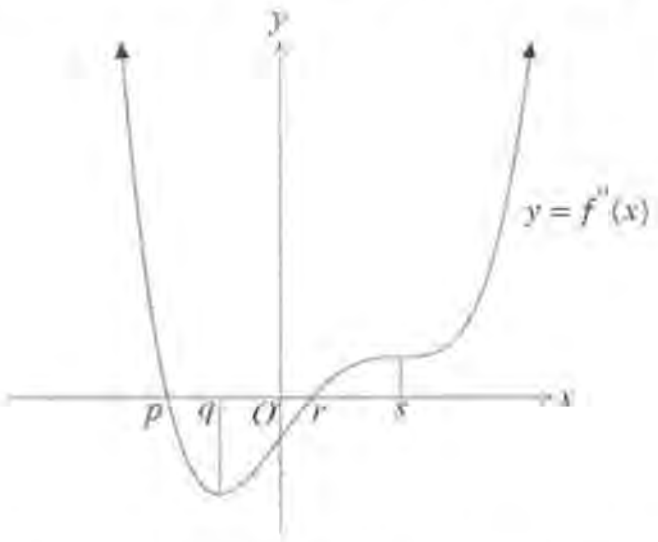
A $t = 0$

B $t = 9$

C $t = 6$

D $t = 3$

5 The diagram shows the graph of $y = f''(x)$ for the function $f(x)$.



For what value of x does the function $f'(x)$ have a maximum turning point?

A $x = p$

B $x = q$

C $x = r$

D $x = s$

6 The table below shows the probability distribution of a discrete random variable X which has mean $-0 \cdot 45$.

x	-2	-1	0	1	2
$P(X = x)$	$0 \cdot 1$	a	$0 \cdot 2$	$0 \cdot 15$	b

What are the values of a and b ?

A $a = 0 \cdot 2,\ b = 0 \cdot 35$

B $a = 0 \cdot 3,\ b = 0 \cdot 25$

C $a = 0 \cdot 4,\ b = 0 \cdot 15$

D $a = 0 \cdot 5,\ b = 0 \cdot 05$

7 The graph of the function $y = f(x)$ is known to have a minimum turning point at $P(6, -4)$. Therefore the graph of $y = -f(-2x)$ will have a maximum turning point at

A $(3, 4)$

B $(-3, 4)$

C $(-3, -4)$

D $(-12, 4)$

8 Let X and Y be two events such that $P(X) = 0 \cdot 5$, $P(Y) = 0 \cdot 6$, and $P(Y|X) = 0 \cdot 7$.

Which of the following statements is FALSE?

A $P(X|Y) < P(Y|X)$

B X and Y are independent events

C $P(X \cap Y) = 0 \cdot 35$

D $P(X \cup Y) = 0 \cdot 75$

9 What are the domain and range of the function $f(x) = \ln(x+1) - \sqrt{4 - x^2}$?

A domain $(-1, \infty)$ range $(-\infty, \infty)$

B domain $(-1, \infty)$ range $(-\infty, \ln 3]$

C domain $(-1, 2]$ range $(-\infty, \infty)$

D domain $(-1, 2]$ range $(-\infty, \ln 3]$

10 Which of the following is the correct function value at the minimum turning point of

$f(x) = (x - 2021)(x - 2022)(x - 2023)$

A $\frac{1}{\sqrt{3}}$

B $-\frac{1}{\sqrt{3}}$

C $-\frac{2\sqrt{3}}{9}$

D $\frac{2\sqrt{3}}{9}$

Question 11 (3 marks)

The third term of an arithmetic series is 32 and the sixth term is 17.

(i) Find the common difference **1**

(ii) Find the sum of the first ten terms. **2**

Question 12 (2 marks)

If $\sin\theta = \frac{3}{5}$ and $\tan\theta < 0$, find the exact value of $\cos\theta$ **2**

Question 13 (3 marks)

Given that $2log_e(x^2y) = 3 + log_e x - log_e y$, express *y* in terms of *x* in simplest terms. **3**

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Question 14 (3 marks)

Find the equation of the normal to $y = e^{\cos x}$ at the point where $x = \frac{\pi}{2}$ **3**

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Question 15 (7 marks)

(i) Find $\int sec^2 3x \; dx$ **1**

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(ii) Find $\int \frac{1}{(3x+2)^4} dx$ **2**

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(iii) Evaluate $\int_0^1 e^{-2x} \, dx$ exactly. **2**

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(iv) Find $\int \frac{x^2}{x^3-5}\ dx$ **2**

Question 16 (3 marks)

The curve $y = \sin x$ is stretched horizontally by a factor of 2, then it is shifted $\frac{\pi}{2}$ units right, then it is stretched vertically by a factor of 3 and reflected in the x-axis.

What equation describes the final curve after this sequence of transformations? **3**

Question 17 (4 marks)

A new brand of electric bicycle is introduced to the market and 18,000 are sold in the first month. Each month thereafter, the sales are 70% of the sales in the previous month.

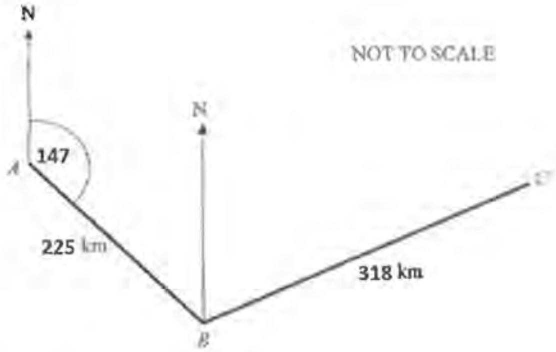
(i) In which month will monthly sales first drop below 1000 per month? **2**

(ii) How many bicycles are sold in total in the first year? **1**

(iii) How many bicycles are eventually sold altogether? **1**

Question 18 (5 marks)

A ship sails 225 km from Adhiban Island on a bearing of 147 degrees and arrive at Port Bologan to pick up some cattle. It then progresses to its destination, Port Cramling, a distance of 318 km on a bearing of 071 degrees.



(i) Show that $\angle ABC = 104°$ **1**
(you may write on the diagram above)

(ii) Show that the distance AC is approximately $431 \cdot 7$ km. **2**

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(iii) The return trip is a straight line back to Adhiban Island and not passing through Port Bologan. Find the bearing that the ship must take to go straight from Port Cramling to Adhiban Island. **2**

Question 19 (5 marks)

Mischa likes to drink pearl milk tea at work. The number X of teas she drinks each day is a random variable with probability distribution given by:

x	0	1	2	3
$P(X = x)$	0.1	0.2	0.3	0.4

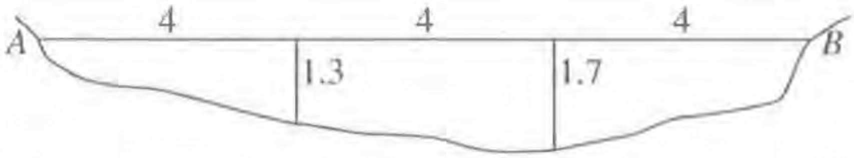
(i) What is the expected value E(X) ? **1**

(ii) What are the variance, Var(X), and the standard deviation, σ ? **2**

(iii) Mischa is at work on two successive days. What is the probability that she drinks the same number of pearl milk teas on both days? **2**

Question 20 (4 marks)

The diagram below shows the cross-section of a stream with the depths of the stream shown in metres at 4 metre intervals. The creek is 12 metres wide.



(i) Use the trapezoidal rule to approximate the area of the cross-section. **2**

(ii) If water flows through this part of the stream at a speed of $0 \cdot 5$ metres/sec, calculate the approximate volume of water that flows past this section in 1 hour. **2**

Question 21 (8 marks)

Consider the function $y = x^3 - 9x^2 + 24x$.

(i) Find all stationary points and determine their nature. **4**

(ii) Find the point of inflection. **2**

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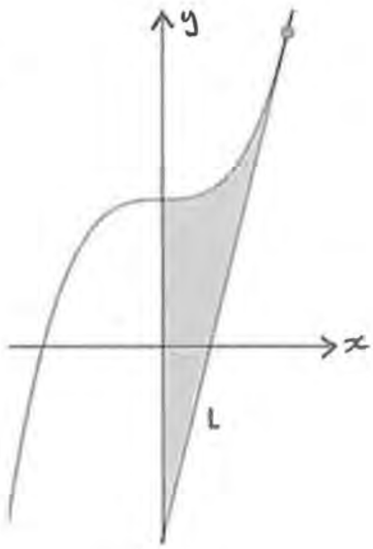
(iii) Sketch the curve showing all important features. **2**

Question 22 (5 marks)

The line **L** is the tangent to the curve $y = x^3 + 7$ at $x = 2$.



(i) Show that the equation of the tangent **L** is $y = 12x - 9$ **2**

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(ii) Find the area bounded by the y-axis, the tangent **L**, and the curve $y = x^3 + 7$ **3**

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Question 23 (5 marks)

(i) Show that $\frac{d}{dx}(x \ln x - x) = \ln x$ **1**

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(ii) Hence or otherwise find $\int \ln x^2 \, dx$ **1**

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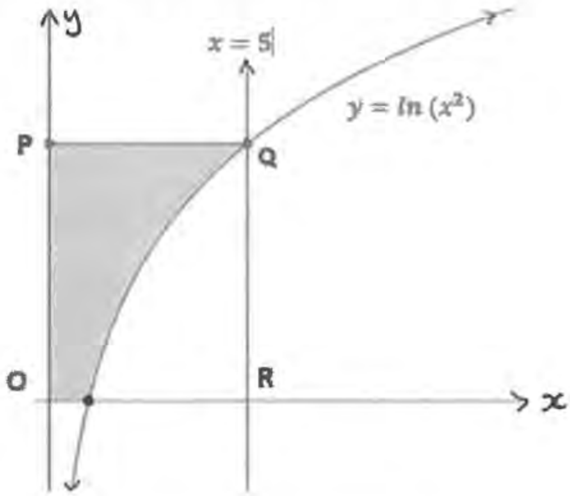
(iii) The graph shows the curve $y = \ln x^2$, $(x > 0)$ which meets the line $x = 5$ at Q. **3**

Using your answers from parts (i) and (ii), or otherwise,

find the area of the shaded region.



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Question 24 (5 marks)

A six-sided die is biased so that the number 5 occurs twice as often as any other number.

(i) The die is rolled once. Show that the probability that an odd number occurs $\frac{4}{7}$. **1**

(ii) If the biased die is rolled twice, find the probability that the sum of the uppermost numbers is seven. **2**

The biased die is now rolled together with TWO fair six-sided dice.

(iii) What is the chance that at least two odd numbers are uppermost? **2**

Question 25 (6 marks)

The outside temperature (in degrees Celsius) on a certain day was modelled by $T = 12 + 7\sin(\frac{\pi t}{12})$ where t is the number of hours after 6am.

(i) What is the maximum temperature in the day? **1**

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(ii) Sketch a graph of the function $T = 12 + 7\sin(\frac{\pi t}{12})$ for $0 \leq t \leq 24$ **2**

(iii) Between what times during the day is the temperature $15°C$ or above? **3**

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Question 26 (4 marks)

Mrs McCrone walks her three labradoodles at Balmoral Beach every Saturday morning. The dogs are poorly behaved: if a stranger pats them, the chance that the white dog bites him is $\frac{1}{20}$, the chance that the brown one bites him is $\frac{1}{10}$, and the chance that the deranged black one bites him is $\frac{1}{2}$.

(i) If a stranger selects one of the dogs at random and pats it, what is the chance they will be bitten? **2**

(ii) Given that a stranger pats one of the dogs and is bitten, what is the probability that it was the black one they patted? **2**

Question 27 (2 marks)

Edward plays a game in which he has a probability p of winning, probability q of losing, and probability r of moving to the next round $(p + q + r = 1)$.

What is his probability of eventually winning, in terms of p and q ? **2**

Question 28 (4 marks)

The point $A(6, 1)$ lies on $h(x)$. The tangent at A is $y = \frac{x}{6}$. Point B is the image of A on the function $g(x) = 3\,h(2x + 4)$.

(i) Show that B has coordinates $(1, 3)$. **1**

(ii) Hence find the equation of the tangent to $g(x)$ at B. **3**

Question 29 (5 marks)

Consider the function $f(x) = \frac{\ln x}{x}$ for $x > 0$.

(i) Show that the graph of $y = f(x)$ has a stationary point at $x = e$. **2**

(ii) By considering the gradient on either side of $x = e$, or otherwise, show that the stationary point at $x = e$ is a maximum. **1**

(iii) Hence deduce that $e^x \geq x^e$ for all $x > 0$. **2**

Question 30 (7 marks)

A truck is making a 1000 kilometre trip at a constant speed of v km/h.

When travelling at v km/h, the truck uses fuel at a rate of $(6 + \frac{v^2}{50})$ litres per hour.

The truck company pays $\$2.00$ per litre for fuel and pays each of the two drivers $\$35$ per hour while the truck is travelling.

(i) Let the total cost of fuel and the driver's pay for the trip be C dollars.
Show that $C = \frac{82000}{v} + 40v$ **3**

(ii) The truck must take no longer than 12 hours to complete the trip, and speed limits require that $v \leq 110$.
At what speed v should the truck travel to minimise the cost C? **4**
(you may disregard any change-over time for the drivers to swap).

2023 - Y12 Advanced Maths - Task 3

Solutions

Section I

1	2	3	4	5	6	7	8	9	10
C	C	A	D	A	D	B	B	D	C

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A $[0, \frac{3}{2}]$

B $\left(0, \frac{3}{2}\right)$

C $(0, 1)$ ✓

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3 Which expression is the derivative of $cos^2 3x$ when differentiated with respect to *x*?

A $-6\sin 3x \cos 3x$ ✓

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4 The amount M of a drug present in the blood after t hours is given by

$M = 9t^2 - t^3$ for $0 \le t \le 9$.

When is the amount of drug in the blood increasing most rapidly?

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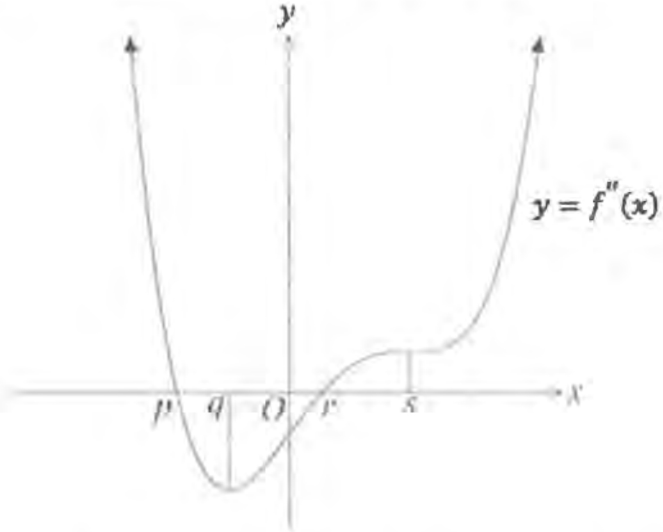
5 The diagram shows the graph of $y = f''(x)$ for the function $f(x)$.



For what value of x does the function $f'(x)$ have a maximum turning point?

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6 The table below shows the probability distribution of a discrete random variable X which has mean $-0 \cdot 45$.

x	-2	-1	0	1	2
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What are the values of a and b ?

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B $a = 0 \cdot 3,\ b = 0 \cdot 25$

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A $(3, 4)$

B $(-3, 4)$ ✓

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8 Let X and Y be two events such that $P(X) = 0 \cdot 5,\ P(Y) = 0 \cdot 6,$ and $P(Y|X) = 0 \cdot 7.$

Which of the following statements is FALSE?

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A domain $(-1, \infty)$ range $(-\infty, \infty)$

B domain $(-1, \infty)$ range $(-\infty, \ln 3]$

C domain $(-1, 2]$ range $(-\infty, \infty)$

D domain $(-1, 2]$ range $(-\infty, \ln 3]$ ✓

10 Which of the following is the correct function value at the minimum turning point of

$f(x) = (x - 2021)(x - 2022)(x - 2023)$

A $\frac{1}{\sqrt{3}}$

B $-\frac{1}{\sqrt{3}}$

C $-\frac{2\sqrt{3}}{9}$ ✓

D $\frac{2\sqrt{3}}{9}$

2023 - Y12 Advanced Task 3
Multiple Choice

① $f(x) = 2\tan\left(4x - \frac{\pi}{3}\right)$

$\therefore$ period $= \frac{\pi}{4}$ $\therefore$ C

② $f(x) = x^4 - 2x^3 \therefore f'(x) = 4x^3 - 6x^2$

$\therefore f''(x) = 12x^2 - 12x = 12x(x-1)$

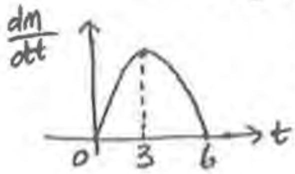
$\therefore 0 < x < 1$ $\therefore$ C

③ $y = (\cos 3x)^2 \therefore y' = 2\cos 3x \cdot -3\sin 3x$

$= -6\sin 3x \cos 3x$ $\therefore$ A

④ $M = 9t^2 - t^3 \therefore \frac{dM}{dt} = 18t - 3t^2$

$= 3t(6-t)$



$\therefore$ at $t = 3$ $\therefore$ D

⑤ Graph is $y = f''(x)$. For $f'(x)$ to have max. turning point, we want $f''(x) = 0$ and f'' changing from + to − $\therefore x = p$ $\therefore$ A

⑥ Probs. add to 1, $\therefore a + b = 0.55$ ---(1)

$E(X) = -0.45 \therefore -0.2 - a + 0 + 0.15 + 2b = -0.45$

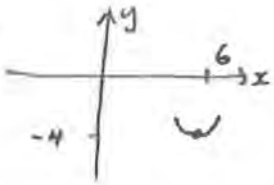
i.e. $-a + 2b = -0.40$ ...(2)

① + ② → $3b = 0.15$

$\therefore b = 0.05$

$a = 0.5$ $\therefore$ D

(7)



$y = f(x)$

Curve has been

- compressed by factor of 2 $\rightarrow f(2x)$
- reflected across y axis $\rightarrow f(-2x)$
- reflected across x axis $\rightarrow -f(-2x)$

$\therefore (6,-4) \rightarrow (3,-4) \rightarrow (-3,-4) \rightarrow (-3,4)$ $\quad \therefore$ (B)

(8) $P(Y|X) = \dfrac{P(Y \cap X)}{P(X)}$ $\quad \therefore 0.7 = \dfrac{P(Y \cap X)}{0.5}$

$\therefore P(Y \cap X) = 0.35$

But $P(Y) \cdot P(X) = (0.6)(0.5) = 0.30$
$\neq P(Y \cap X)$

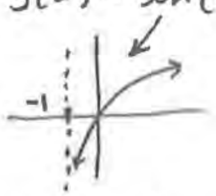
$\therefore$ <u>not</u> independent $\quad \therefore$ (B)

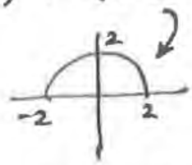
[<u>Note</u> : $P(X|Y) = \dfrac{P(X \cap Y)}{P(Y)} = \dfrac{0.35}{0.6} \doteqdot 0.58 < P(Y|X)$ ✓

$P(X \cap Y) = 0.35$ ✓

$P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$
$= 0.5 + 0.6 - 0.35 = 0.75$ ✓]

(9) $f(x) = \ln(x+1) - \sqrt{4-x^2}$





$D: (-1, \infty)$ $\qquad D: [-2,2] \longrightarrow D: (-1,2]$ $\quad \therefore$ (D)

$R: (-\infty, \infty)$ $\qquad R: [0,2] \longrightarrow R: (-\infty, \ln 3]$

(10) $f(x) = (x-2021)(x-2022)(x-2023)$ is
just a horizontal translation of
$f(x) = (x+1)(x)(x-1)$, & so will have same
y values at min. Turning point. Let $f(x) = x(x^2-1) = x^3 - x$

$f'(x) = 3x^2 - 1 = 0$ when $x = -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

$f''(x) = 6x > 0$ when $x = \frac{1}{\sqrt{3}}$

$\therefore y = \left(\frac{1}{\sqrt{3}} + 1\right)\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}} - 1\right) = \frac{1}{\sqrt{3}}\left(\frac{1}{3} - 1\right)$ $\quad \therefore$ (C)

$= \dfrac{-2}{3\sqrt{3}} = \dfrac{-2\sqrt{3}}{9}$

Question 11 (3 marks)

The third term of an arithmetic series is 32 and the sixth term is 17.

(i) Find the common difference **1**

$T_3 = a + 2d = 32 \quad ---- (1)$

$T_6 = a + 5d = 17 \quad ---- (2)$

$(1) - (2) \rightarrow \quad -3d = 15 \qquad \therefore d = -5$ ✓

(ii) Find the sum of the first ten terms. **2**

From (i), $a = 32 - 2(-5) \quad \therefore a = 42$

$\therefore \quad S_{10} = \frac{10}{2}\left(2(42) + (10-1)(-5)\right)$

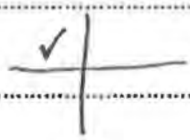
$= 5\,(84 - 45)$

$\therefore \quad S_{10} = 195$ ✓✓

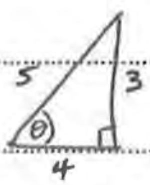
Question 12 (2 marks)

If $\sin\theta = \frac{3}{5}$ and $\tan\theta < 0$, find the exact value of $\cos\theta$ **2**

$\sin > 0$, $\tan < 0 \quad \therefore \quad Q2$



$\therefore \cos < 0$



$\therefore \cos\theta = -\frac{4}{5}$ ✓✓

Question 13 (3 marks)

Given that $2log_e(x^2y) = 3 + log_e x - log_e y$, express y in terms of x in simplest terms. **3**

$2\ln(x^2y) = 3 + \ln x - \ln y$

$\therefore 2\ln x^2 + 2\ln y = 3 + \ln x - \ln y$

$4\ln x + 2\ln y = 3 + \ln x - \ln y$

$3\ln x + 3\ln y = 3$

$\ln x + \ln y = 1$

$\ln(xy) = 1$

$\therefore xy = e$

$\therefore y = \frac{e}{x}$ ✓

Correctly uses log laws ✓✓

ALT:

$\ln x^4y^2 = 3 + \ln x - \ln y$

$\ln x^4y^2 - \ln x + \ln y = 3$

$\ln \frac{x^4y^2 \times y}{x} = 3$

$\ln(x^3y^3) = 3$

$3\ln xy = 3$

$\ln xy = 1$

$xy = e$

$y = \frac{e}{x}$

Question 14 (3 marks)

Find the equation of the normal to $y = e^{\cos x}$ at the point where $x = \frac{\pi}{2}$ **3**

$y' = -\sin x \cdot e^{\cos x}$ ✓

When $x = \frac{\pi}{2}$, $y = e^{\cos \frac{\pi}{2}} = e^0 = 1$

& $y' = -\sin\frac{\pi}{2} \cdot e^{\cos\frac{\pi}{2}} = -1$

$\therefore m_N = 1$ ✓

So normal is $y - 1 = 1\left(x - \frac{\pi}{2}\right)$

$\therefore y = x + 1 - \frac{\pi}{2}$ ✓

$\left(\text{ie } x - y + 1 - \frac{\pi}{2} = 0\right)$

Question 15 (7 marks)

(i) Find $\int sec^2 3x \; dx$ **1**

$= \frac{1}{3} \tan 3x + C$

(Note: maximum penalty 1 mark for lack of constant)

(ii) Find $\int \frac{1}{(3x+2)^4} dx$ **2**

$\int (3x+2)^{-4} dx = \frac{(3x+2)^{-3}}{-3 \times 3} + C$

$\therefore = \frac{-1}{9(3x+2)^3} + C$

(iii) Evaluate $\int_0^1 e^{-2x} \, dx$ exactly. **2**

$\int_0^1 e^{-2x} dx = \left[-\frac{e^{-2x}}{2} \right]_0^1$

$= -\frac{e^{-2}}{2} - \left(-\frac{1}{2}\right)$

$\therefore = \frac{1}{2} - \frac{1}{2e^2}$

$\left(\text{ie } \frac{e^2-1}{2e^2}\right)$

(iv) Find $\int \frac{x^2}{x^3-5}\ dx$ **2**

$$= \frac{1}{3}\int \frac{3x^2}{x^3-5}\ dx$$

$$= \frac{1}{3}\ln\left|x^3-5\right| + C$$

✓ Correct answer

✓ absolute value

Question 16 (3 marks)

The curve $y = \sin x$ is stretched horizontally by a factor of 2, then it is shifted $\frac{\pi}{2}$ units right, then it is stretched vertically by a factor of 3 and reflected in the x-axis.

What equation describes the final curve after this sequence of transformations? **3**

$$y = \sin x \longrightarrow y = \sin\frac{x}{2}$$

✓ horiz. stretch.

$$\longrightarrow y = \sin\frac{1}{2}\left(x - \frac{\pi}{2}\right)$$

✓ shift R

$$\rightarrow y = -3\sin\frac{1}{2}\left(x - \frac{\pi}{2}\right)$$

✓ vert. stretch & reflection

$$\left(\underline{\text{ie}}\quad y = -3\sin\left(\frac{x}{2} - \frac{\pi}{4}\right)\right)$$

Question 17 (4 marks)

A new brand of electric bicycle is introduced to the market and 18,000 are sold in the first month. Each month thereafter, the sales are 70% of the sales in the previous month.

(i) In which month will monthly sales first drop below 1000 per month? **2**

we want $T_n = 18000 \times (0.7)^{n-1} < 1000$

$\therefore 0.7^{n-1} < \frac{1}{18}$

$(n-1)\log_{10} 0.7 < \log_{10} \frac{1}{18}$

$n-1 > \frac{\log_{10} \frac{1}{18}}{\log_{10} 0.7}$ $\therefore n > 8.10365 + 1$

$n > 9.1$

$\therefore$ in the 10^{th} month.

(ii) How many bicycles are sold in total in the first year? **1**

$S_{12} = \frac{18000(1-0.07^{12})}{1-0.07} \doteqdot 59169.52\ldots$

$\therefore$ 59169 sold in 1^{st} year

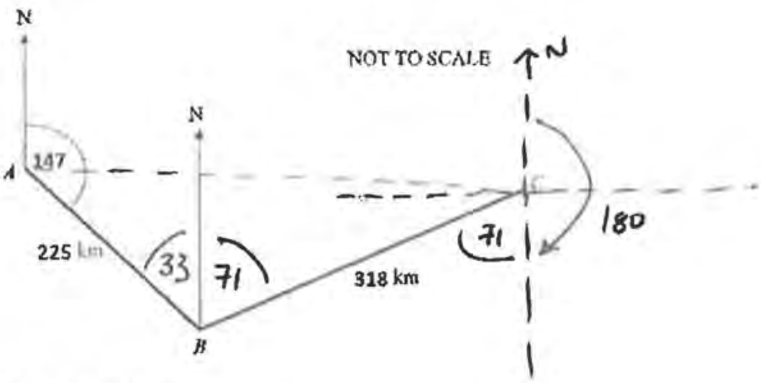
(iii) How many bicycles are eventually sold altogether? **1**

as $|r|<1$ $\therefore$ limiting sum exists

$\therefore S_\infty = \frac{18000}{1-0.07} = 60000$

Question 18 (5 marks)

A ship sails 225 km from Adhiban Island on a bearing of 147 degrees and arrive at Port Bologan to pick up some cattle. It then progresses to its destination, Port Cramling, a distance of 318 km on a bearing of 071 degrees.



(i) Show that $\angle ABC = 104°$ **1**
(you may write on the diagram above)

$\angle ABC = (180° - 147°) + 71° = 33° + 71° = 104°.$

alternate angles in parallel lines

bearing of C from B

(ii) Show that the distance AC is approximately $431 \cdot 7$ km. **2**

$AC^2 = 225^2 + 318^2 - 2(225)(318)\cos 104°$

$\doteqdot 186\,368 \cdot 0233\ldots$

$\therefore AC \doteqdot 431 \cdot 7036\ldots$

Thus $AC = 431 \cdot 7$ km (1d.p.)

(iii) The return trip is a straight line back to Adhiban Island and not passing through Port Bologan. Find the bearing that the ship must take to go straight from Port Cramling to Adhiban Island. **2**

$$\frac{\sin \angle ACB}{225} = \frac{\sin 104}{431.7036}$$

$\therefore \sin \angle ACB \doteqdot 0.505709296\ldots$

$\therefore \angle ACB \doteqdot 30^\circ 23'$

$\therefore$ Bearing $= 180^\circ + 71^\circ + 30^\circ 23'$

$= 281^\circ 23'$ T.

Question 19 (5 marks)

Mischa likes to drink pearl milk tea at work. The number *X* of teas she drinks each day is a random variable with probability distribution given by:

x	0	1	2	3
$P(X = x)$	0.1	0.2	0.3	0.4

(i) What is the expected value E(X) ? **1**

$E(X) = 0(0.1) + 1(0.2) + 2(0.3) + 3(0.4)$

$= 0 + 0.2 + 0.6 + 1.2 = 2$

(ii) What are the variance, Var(X), and the standard deviation, σ ? **2**

$Var(X) = E(X^2) - \mu^2$

$= (0.1)(0^2) + 0.2(1^2) + 0.3(2^2) + 0.4(3^2) - 2^2$

$= 5 - 4$

$\therefore Var(X) = 1, \quad \sigma = \sqrt{1} = 1$

(iii) Mischa is at work on two successive days. What is the probability that she drinks the same number of pearl milk teas on both days? **2**

probability $= P(0,0) + P(1,1) + P(2,2) + P(3,3)$

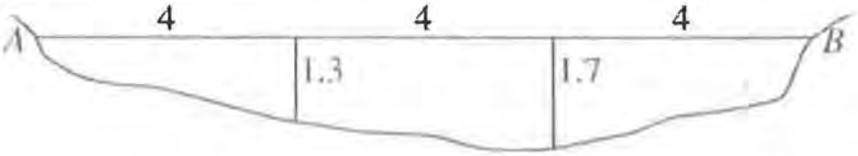
$= (0.1)^2 + (0.2)^2 + (0.3)^2 + (0.4)^2$ ✓

$= 0.01 + 0.04 + 0.09 + 0.16$

$\therefore$ probability $= 0.3$ ✓

Question 20 (4 marks)

The diagram below shows the cross-section of a stream with the depths of the stream shown in metres at 4 metre intervals. The creek is 12 metres wide.



(i) Use the trapezoidal rule to approximate the area of the cross-section. **2**

x	0	4	8	12
y	0	1.3	1.7	0

$A \doteqdot \frac{12-0}{2\times 3}\left[0+0+2(1.3+1.7)\right] = 12\ m^2$ ✓ ✓

(ii) If water flows through this part of the stream at a speed of $0 \cdot 5$ metres/sec, calculate the approximate volume of water that flows past this section in 1 hour. **2**

$V = A \times \text{length}$

length = speed × time

$= 0.5 \times (60\times 60)\,m$

$= 1800\ m$ ✓

$\therefore V \doteqdot 12 \times 1800\ m^3$

$V \doteqdot 21\,600\ m^3$

$(\doteqdot 21600\ kL)$ ✓

Question 21 (8 marks)

Consider the function $y = x^3 - 9x^2 + 24x$.

(i) Find all stationary points and determine their nature. **4**

$y = x^3 - 9x^2 + 24x$

$\therefore y' = 3x^2 - 18x + 24$

$= 3(x^2 - 6x + 8)$

$y' = 3(x-4)(x-2)$

For stationary points, $y' = 0$ $\therefore (x-4)(x-2) = 0$

$\therefore x = 4$ or $x = 2$ ✓ $x=4,\ x=2$.

$y = 20$ $y = 16$

$y'' = 6x - 18 = 6(x-3)$

✓ for testing.

At $(4,20)$, $y'' = 6(4-3) = 6 > 0$

$\therefore$ minimum turning point at $(4,20)$ ✓

min. point

At $(2,16)$, $y'' = 6(2-3) = -6 < 0$

$\therefore$ maximum turning point at $(2,16)$. ✓

max. point.

(ii) Find the point of inflection. **2**

For inflections, $y'' = 0$ $\therefore 6(x-3) = 0$

$\therefore x = 3, \; y = 18$ ✓ point

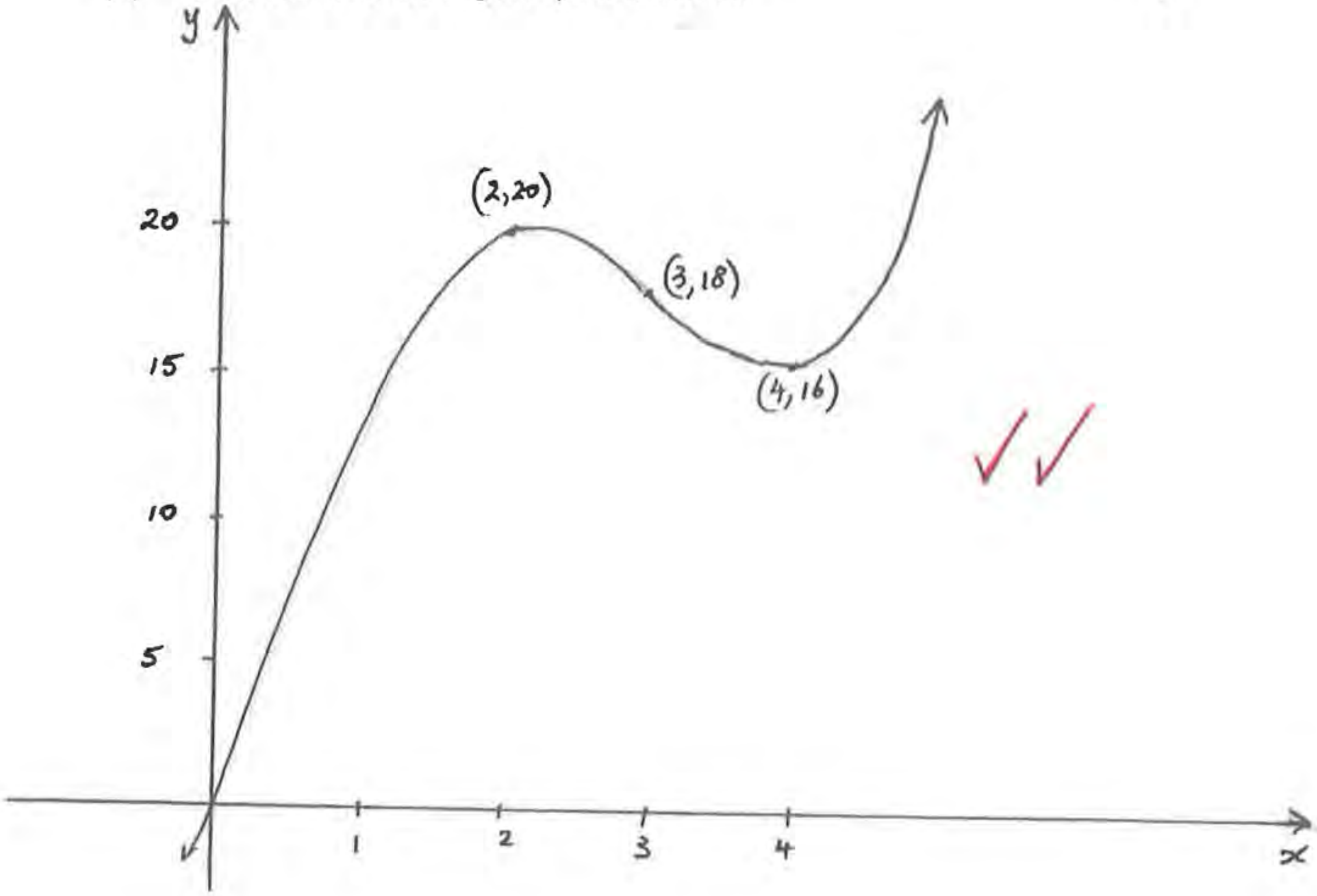
✓ testing.

x	2	3	4
y''	-6	0	6

$\therefore$ change in concavity at $x = 3$

$\therefore (3, 18)$ is point of inflection.

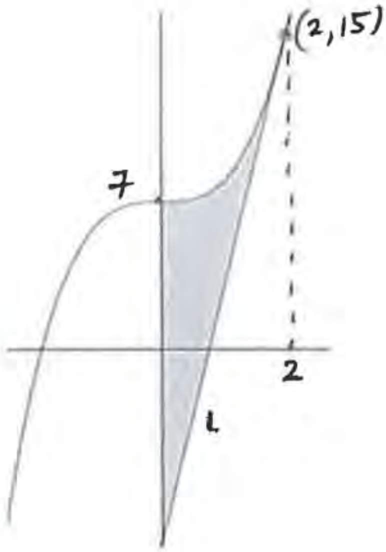
(iii) Sketch the curve showing all important features. **2**



✓✓

Question 22 (5 marks)

The line **L** is the tangent to the curve $y = x^3 + 7$ at $x = 2$.



(i) Show that the equation of the tangent **L** is $y = 12x - 9$ **2**

$y' = 3x^2$

$\therefore$ At $x=2$, $y' = 12$ and $y = 15$

$\therefore$ tangent equation is

$y - 15 = 12(x-2)$

$y - 15 = 12x - 24$

$\therefore$ $y = 12x - 9$, as required.

(ii) Find the area bounded by the y-axis, the tangent **L**, and the curve $y = x^3 + 7$ **3**

$$A = \int_0^2 \left[x^3 + 7 - (12x - 9)\right] dx$$

$$= \int_0^2 (x^3 - 12x + 16)\, dx$$

$$= \left[\frac{x^4}{4} - 6x^2 + 16x\right]_0^2$$

$$= \left[\frac{16}{4} - 6(4) + 16(2)\right] - \left[0 - 0 + 0\right]$$

$\therefore$ $A = 12u^2$

Question 23 (5 marks)

(i) Show that $\frac{d}{dx}(x \ln x - x) = \ln x$ **1**

$\frac{d}{dx}[x \ln x - x] = \ln x \cdot 1 + x \cdot \frac{1}{x} - 1 = \ln x + 1 - 1$

$= \ln x$, as required.

(ii) Hence or otherwise find $\int \ln x^2 \, dx$ **1**

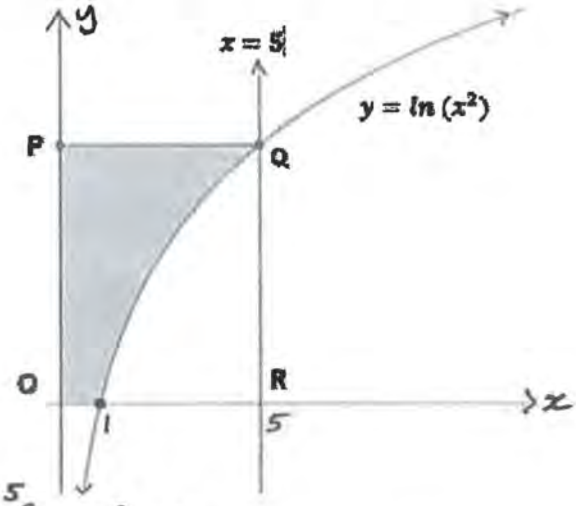
$\ln x^2 = 2 \ln x$

$\therefore \int \ln x^2 \, dx = 2(x \ln x - x) + C$, from (i).

(iii) The graph shows the curve $y = \ln x^2$, $(x > 0)$ which meets the line $x = 5$ at Q. **3**

Using your answers from parts (i) and (ii), or otherwise,

find the area of the shaded region.



Area of rectangle OPQR

$= 5 \times \ln 25$

$= 5 \times \ln 5^2$

$\therefore$ Area OPQR $= 10 \ln 5 \ u^2$.

$\therefore$ Area shaded region $= 10 \ln 5 - \int_1^5 \ln x^2 \, dx$

Thus Area $= 10 \ln 5 - 2\left[x \ln x - x\right]_1^5$

$= 10 \ln 5 - 2\left[(5 \ln 5 - 5) - (\ln 1 - 1)\right]$

$= 10 \ln 5 - (10 \ln 5 - 8)$

$\therefore$ Area $= 8 \ u^2$

Question 24 (5 marks)

A six-sided die is biased so that the number 5 occurs twice as often as any other number.

(i) The die is rolled once. Show that the probability that an odd number occurs $\frac{4}{7}$. **1**

Sample space is $\{1, 2, 3, 4, 5, 5, 6\}$

(these elements are equally likely : we give two

5's since it is twice as likely as the others)

Event is $\{1, 3, 5, 5\}$.

$\therefore P(odd) = \frac{4}{7}$

(ii) If the biased die is rolled twice, find the probability that the sum of the uppermost numbers is seven. **2**

$E = \{(2,5), (5,2), (3,4), (4,3), (1,6), (6,1)\}$

$\therefore P(E) = \frac{1}{7} \times \frac{2}{7} + \frac{2}{7} \times \frac{1}{7} + 4 \times \left(\frac{1}{7} \times \frac{1}{7}\right)$

$= \frac{8}{49}$

The biased die is now rolled together with TWO fair six-sided dice.

(iii) What is the chance that at least two odd numbers are uppermost? **2**

"At least 2 odd numbers" means

"3 odd numbers" or "2 odd and 1 even".

Now,

$P(\text{odd number on fair die}) = \frac{1}{2}$, $P(\text{odd n° on biassed die}) = \frac{4}{7}$

$P(\text{even number on fair die}) = \frac{1}{2}$, $P(\text{even n° on biased die}) = \frac{3}{7}$

biased die

biased die

$\therefore P(E) = \frac{1}{2} \times \frac{1}{2} \times \frac{4}{7} + P\left[(O,O,E), \text{ or } (O,E,O) \text{ or } (E,O,O)\right]$

$= \frac{1}{7} + \frac{4}{7} \times \frac{1}{2} \times \frac{1}{2} + \frac{4}{7} \times \frac{1}{2} \times \frac{1}{2} + \frac{3}{7} \times \frac{1}{2} \times \frac{1}{2}$

$= \frac{1}{7} + \frac{11}{28}$

$P(E) = \frac{15}{28}$ ✓✓

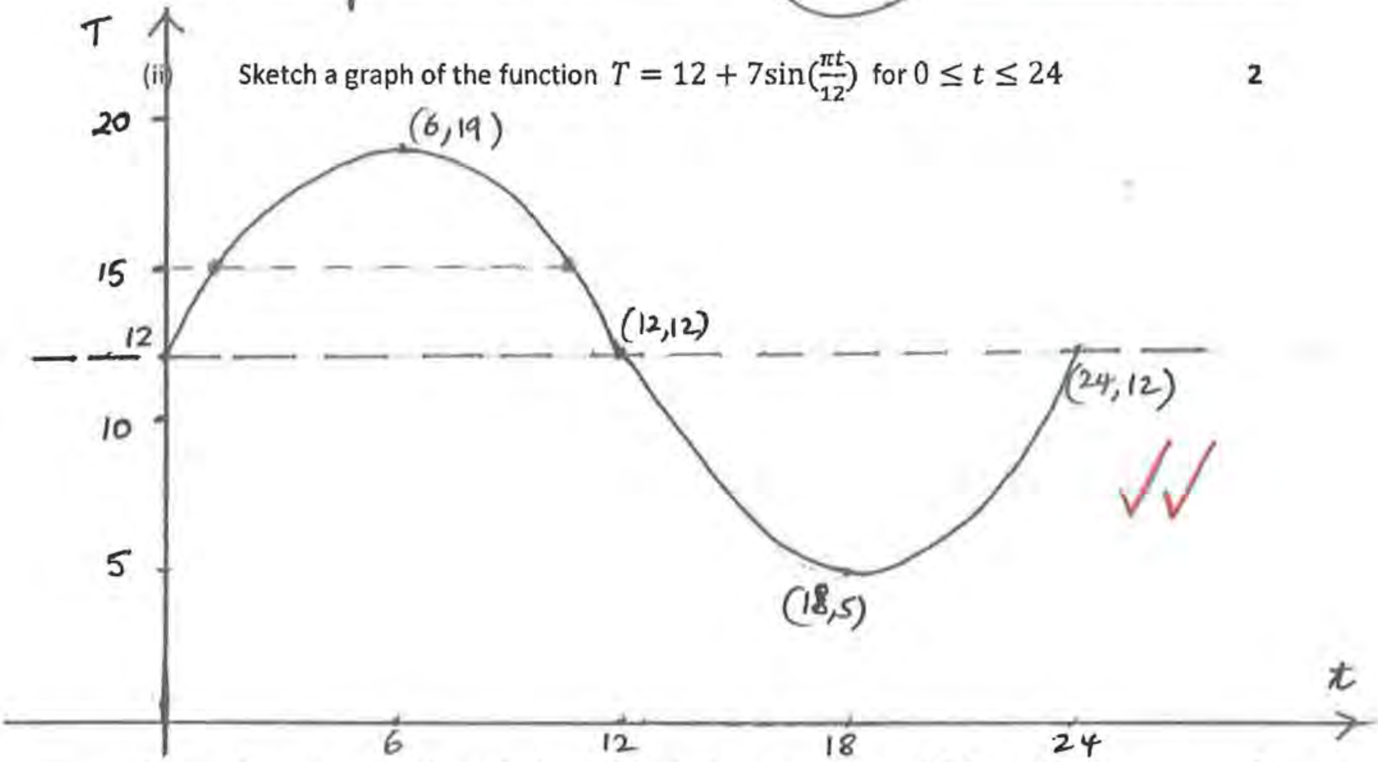
Question 25 (6 marks)

The outside temperature (in degrees Celsius) on a certain day was modelled by $T = 12 + 7\sin(\frac{\pi t}{12})$ where t is the number of hours after 6am.

(i) What is the maximum temperature in the day? **1**

Max. temp. $= 12 + 7 = 19°C$

(ii) Sketch a graph of the function $T = 12 + 7\sin(\frac{\pi t}{12})$ for $0 \le t \le 24$ **2**



(iii) Between what times during the day is the temperature $15°C$ or above? **3**

Let $12 + 7\sin\left(\frac{\pi t}{12}\right) = 15$

$\therefore \sin\left(\frac{\pi t}{12}\right) = \frac{3}{7}$

Thus $t = \frac{12}{\pi}\sin^{-1}\frac{3}{7}$ or $t = \frac{12}{\pi}\left(\pi - \sin^{-1}\frac{3}{7}\right)$

$\therefore t \doteqdot 1.692$ hours or 10.308 hours

ie times are 7.692 or 16.308

$\therefore \geqslant 15°C$ from 7:42am and 16:18 pm.

Question 26 (4 marks)

Mrs McCrone walks her three labradoodles at Balmoral Beach every Saturday morning. The dogs are poorly behaved: if a stranger pats them, the chance that the white dog bites him is $\frac{1}{20}$, the chance that the brown one bites him is $\frac{1}{10}$, and the chance that the deranged black one bites him is $\frac{1}{2}$.

(i) If a stranger selects one of the dogs at random and pats it, what is the chance they will be bitten? **2**

$$P(\text{bitten}) = \frac{1}{3} \times \frac{1}{20} + \frac{1}{3} \times \frac{1}{10} + \frac{1}{3} \times \frac{1}{2} \checkmark$$

$$= \frac{13}{60} \checkmark$$

(ii) Given that a stranger pats one of the dogs and is bitten, what is the probability that it was the black one they patted? **2**

$$P(\text{Black} \mid \text{bitten}) = \frac{P(\text{Black and bitten})}{P(\text{bitten})}$$

$$= \frac{\frac{1}{3} \times \frac{1}{2}}{\frac{13}{60}} \checkmark$$

$$= \frac{10}{13} \checkmark$$

Question 27 (2 marks)

Edward plays a game in which he has a probability p of winning, probability q of losing, and probability r of moving to the next round $(p + q + r = 1)$.

What is his probability of eventually winning, in terms of p and q ? **2**

Let W = 'win round 1', L = 'lose round 1',

C = 'continue after round 1'.

The $P(\text{eventually wins}) = P(\text{eventually wins} \mid W) \cdot P(W)$

$+ P(\text{eventually wins} \mid L) \cdot P(L)$

$+ P(\text{eventually wins} \mid C) \cdot P(C)$

$= 1 \times p + 0 \times q + r \times p(\text{eventually wins})$,

since $p(\text{eventually wins} \mid C) = p(\text{eventually wins})$.

$\therefore P(\text{eventually wins}) = p + r \times p(\text{eventually wins})$

$\therefore p(\text{eventually wins}) = \frac{p}{1-r}$

$= \frac{p}{p+q}$

✓✓

Question 28 (4 marks)

The point $A(6, 1)$ lies on $h(x)$. The tangent at A is $y = \frac{x}{6}$. Point B is the image of A on the function $g(x) = 3\,h(2x + 4)$.

(i) Show that B has coordinates $(1, 3)$. **1**

$g(x) = 3h(2x+4) = 3h[2(x+2)]$

$\therefore$ for any point (x,y) on $h(x)$, the image on

$g(x)$ is $\left(\frac{x}{2} - 2,\ 3y\right)$

$\therefore B = \left(\frac{6}{2} - 2,\ 1 \times 3\right) = (1, 3)$.

✓

(ii) Hence find the equation of the tangent to $g(x)$ at B. 3

Method 1: tangent to $h(x)$ at A is $y = \frac{x}{6}$

$\therefore$ gradient of tangent to $h(x)$ at A is $1/6$

Thus gradient of tangent to $g(x)$ at B is

$m = \frac{1}{6}$: vertically dilated by factor of 3 &

horizontally dilated by factor of $\frac{1}{2}$

$\therefore m_B = \frac{1}{6} \times \frac{3}{\frac{1}{2}} = 1$ ✓✓

$\therefore$ tangent to $g(x)$ at B is $y-3 = 1(x-1)$

i.e. $y = x+2$ (or $x-y+2=0$). ✓

Method 2:

$g(x) = 3h(2x+4)$

$\therefore g'(x) = 3h'(2x+4) \times 2 = 6h'(2x+4)$.

Since $g'(1)$ = gradient of tangent at B,

$\therefore g'(1) = 6h'[2(1)+4]$

$= 6h'(6)$

$= 6 \times$ gradient of tangent to $h(x)$ at A

$= 6 \times \frac{1}{6}$

$= 1$

$\therefore$ tangent to $g(x)$ at B is $y-3 = 1(x-1)$

i.e. $y = x+2$.

Question 29 (5 marks)

Consider the function $f(x) = \frac{\ln x}{x}$ for $x > 0$.

(i) Show that the graph of $y = f(x)$ has a stationary point at $x = e$. **2**

$$y' = \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2}$$

Stationary point means $y' = 0$ $\therefore 1 - \ln x = 0$

$\therefore \ln x = 1$ $\therefore x = e$

So $\therefore$ stationary point at $x = e$.

(ii) By considering the gradient on either side of $x = e$, or otherwise, show that the stationary point at $x = e$ is a maximum. **1**

x	2.71	e	2.72
y'	0.0004	0	-0.00009
	/	—	\

$\therefore$ max. at $x = e$.

[alt. : $y'' = \frac{(x^2)(-\frac{1}{x}) - (1 - \ln x)(2x)}{x^4} = \frac{-3x + 2x\ln x}{x^4}$

$\therefore y''(e) = -0.0498 < 0$ $\therefore$ max at $x = e$]

(iii) Hence deduce that $e^x \geq x^e$ for all $x > 0$. **2**

From (ii), $\frac{\ln x}{x}$ $(x>0)$ has a maximum at $x=e$.

$f(e) = \frac{\ln e}{e} = \frac{1}{e}$

Thus $\frac{\log_e x}{x} \leq \frac{1}{e}$, $x>0$

$\therefore \log_e x \leq \frac{x}{e}$ (as $x>0$)

$e \log_e x \leq x$

$\therefore \log_e x^e \leq x$

$\therefore e^{\log_e x^e} \leq e^x$

$\therefore x^e \leq e^x$ (as $e^{\log_e a} = a$)

i.e. $e^x \geq x^e$ ✓✓

Question 30 (7 marks)

A truck is making a 1000 kilometre trip at a constant speed of v km/h.

When travelling at v km/h, the truck uses fuel at a rate of $(6 + \frac{v^2}{50})$ litres per hour.

The truck company pays $\$2.00$ per litre for fuel and pays each of the two drivers $\$35$ per hour while the truck is travelling.

(i) Let the total cost of fuel and the driver's pay for the trip be C dollars.
Show that $C = \frac{82000}{v} + 40v$ **3**

Time for trip $= \frac{1000}{v}$ hours

$\therefore$ Driver cost $= 2 \times \$35 \times \frac{1000}{v} = \$\frac{70\,000}{v}$

Fuel cost $= \$2 \times \left(6 + \frac{v^2}{50}\right) \times \frac{1000}{v} = \frac{2000}{v}\left(6 + \frac{v^2}{50}\right)$

$= \frac{12000}{v} + 40v$

$\therefore \quad C = \frac{70000}{v} + \frac{12000}{v} + 40v$

$\therefore \quad C = \frac{82000}{v} + 40v$ dollars.

(ii) The truck must take no longer than 12 hours to complete the trip, and speed limits require that $v \leq 110$.
At what speed v should the truck travel to minimise the cost C? **4**
(you may disregard any change-over time for the drivers to swap).

$C = 82000v^{-1} + 40v$

$\therefore \frac{dC}{dv} = -\frac{82000}{v^2} + 40.$

For min, $\frac{dC}{dv} = 0 \quad \therefore \quad \frac{82000}{v^2} = 40 \quad \therefore v^2 = 2050$

$\therefore \quad v = \sqrt{2050} \doteqdot 45.3$ km/h.

But, if $v = \sqrt{2050}$, $t = \frac{1000}{\sqrt{2050}} \doteqdot 22 \cdot 1$ hours

This is too long, as we need $t \leq 12$ (ie. $v \geq 83\frac{1}{3}$)

In fact, the domain here is $83\frac{1}{3} \leq v \leq 110$.

So we need to check the cost at the endpoints of the domain.

If $v = 83\frac{1}{3}$, $C = \frac{82000}{83\frac{1}{3}} + 40(83\frac{1}{3}) \doteqdot \$4317$

If $v = 110$, $C = \frac{82000}{110} + 40(110) \doteqdot \$5145$

So we get minimum cost when $v = 83\frac{1}{3}$ km/h.

[Since $\frac{d^2C}{dv^2} = 2 \times 82000 v^{-3} > 0$ for all $v > 0$

So $v = \sqrt{2050}$ gives min. turning point &

C is increasing after $v = \sqrt{2050}$. As v increases, C increases, so the minimum C that satisfies the conditions corresponds to $v = 83\frac{1}{3}$ km/h.

But all that's required is to calculate C at the endpoints.].

ENDS