HORNSBY GIRLS HIGH SCHOOL

Mathematics Extension 1

Year 12 Higher School Certificate

Trial Examination Term 3 2024

STUDENT NUMBER: ______________________________

TEACHER NAME: Ms Guan, Ms Murray

STUDENT NAME: ______________________________ **(circle one)** Mr Payne, Mrs Sztajer

General Instructions:

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- In Questions 11–14, show relevant mathematical reasoning and/ or calculations

Total Marks: 70

Section I – 10 marks (pages 3–6)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 7–10)

- Attempt Questions 11–14
- Start each question in a new writing booklet
- Write your student number on every writing booklet
- Allow about 1 hour and 45 minutes for this section

Question	*1-10*	*11*	*12*	*13*	*14*	*Total*
Total	*/10*	*/15*	*/15*	*/15*	*/15*	*/70*

Outcomes assessed: ME 12-1, 12-2, 12-3, 12-4, 12-7

Section I

10 marks
Attempt Questions 1 - 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10

Question 1

What is the size of the angle between the vectors $\underset{\sim}{a} = 3\underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} - \underset{\sim}{j}$?

(A) $\frac{\pi}{6}$

(B) $\frac{\pi}{4}$

(C) $\frac{\pi}{3}$

(D) $\frac{\pi}{2}$

Question 2

Which of the following integrals is equivalent to $\int \cos^2 4x \, dx$?

(A) $\int \frac{1-\cos 8x}{2} \, dx$

(B) $\int \frac{1+\cos 8x}{2} \, dx$

(C) $\int \frac{1-\cos 4x}{2} \, dx$

(D) $\int \frac{1+\cos 4x}{2} \, dx$

Question 3

What is the remainder when $P(x) = x^3 - 3x^2 + 2x + 3$ is divided by $(x+1)$?

(A) –3

(B) –2

(C) 2

(D) 3

Question 4

Consider the differential equation $\frac{dy}{dx} = \frac{1}{xy}$.

Which of the following equations best represents this relationship between x and y ?

(A) $y = \frac{1}{x} + c$

(B) $y = \ln|x| + c$

(C) $y^2 = \ln|x| + c$

(D) $y^2 = 2\ln|x| + c$

Question 5

For the two non-zero vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ it is known that $\overrightarrow{OA} \bullet \overrightarrow{OB} = 0$.

Which of the following MUST be true?

(A) Either $\overrightarrow{OA} = 0$ or $\overrightarrow{OB} = 0$

(B) $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are parallel

(C) $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are perpendicular

(D) O, A and B are collinear

Question 6

What is the constant term in the binomial expansion $\left(3x-\frac{2}{x^2}\right)^9$?

(A) $\binom{9}{3}3^3.2^6$

(B) $\binom{9}{6}3^6.2^3$

(C) $-\binom{9}{3}3^3.2^6$

(D) $-\binom{9}{6}3^6.2^3$

Question 7

Which of the following is equivalent to $\sin x+\sqrt{3}\cos x$ expressed in the form $A\cos(x+\theta)$?

(A) $2\cos\left(x-\frac{\pi}{6}\right)$

(B) $2\cos\left(x+\frac{\pi}{6}\right)$

(C) $4\cos\left(x-\frac{\pi}{6}\right)$

(D) $4\cos\left(x+\frac{\pi}{6}\right)$

Question 8

Six adults and four children need to be seated at a circular table. How many arrangements exist if two particular children must be separated?

(A) 10 080

(B) 17 280

(C) 282 240

(D) 362 880

Question 9

Which of the following pairs of parametric equations are **NOT** equivalent to $y=\sqrt{x+1}$?

(A) $x=t^2-1,\ y=t\ \ for\ t\geq 0$

(B) $x=t,\ y=\sqrt{t+1}\ \ for\ t\geq 1$

(C) $x=t-1,\ y=\sqrt{t}\ \ for\ t\geq 0$

(D) $x=t-2,\ y=\sqrt{t-1}\ \ for\ t\geq 1$

Question 10

What is the domain of the function $y=\sin(\arcsin x)$?

(A) $-1\leq x\leq 1$

(B) $-\frac{\pi}{2}\leq x\leq\frac{\pi}{2}$

(C) $0\leq x\leq\pi$

(D) All real x

Section II

60 marks
Attempt Questions 11 to 14
Allow about 1 hour and 45 minutes for this section
Instructions

- Answer the questions in the appropriate writing booklet.
- In Questions 11-14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a new writing booklet.

(a) Find $\left(2\underset{\sim}{i}+3\underset{\sim}{j}\right)-\left(\underset{\sim}{i}-2\underset{\sim}{j}\right)$. **1**

(b) Use the $t=\tan\dfrac{x}{2}$ to show that $\sin x-\tan\dfrac{x}{2}=\tan\dfrac{x}{2}\cos x$. **2**

(c) Use the substitution $u=x+2$ to find $\int x\sqrt{x+2}\,dx$. **3**

(d) Solve the inequality $8-|2x-1|\le 5$. **2**

(e) A cylindrical tank of radius 2 m is being filled with water so that the volume is increasing at a constant rate of 3 m^3/min.

Find the rate of increase of the depth of the water in the tank. **2**

(f) Evaluate $\displaystyle\int_0^1 \frac{dx}{x^2+3}$ **2**

(g) Two swimmers want to swim from a point A on one island to point B on another island where B is due east of A. The ocean has a current of 1.2 ms^{-1} in the direction of 135°T.
Swimmer X swims at 3 ms^{-1} in the direction of 063.43°T in order to reach point B.
Swimmer Y swims at 2.4 ms^{-1}.
On which bearing does swimmer Y need to swim in order to reach point B? **3**

End of Question 11

Question 12 (15 marks) Start a new writing booklet and use the diagrams provided.

(a) Consider function $f(x) = 2^x - 3$,

(i) On the number plane provided on the separate sheet, sketch the graph of $y = f(x)$ showing the x- and y- intercept in exact values. **2**

(ii) Find the equation of the inverse function $f^{-1}(x)$ as a natural logarithm function and state its domain and range. **3**

(iii) On the same number plane for part (i), sketch the graph of $y = f^{-1}(x)$. **1**

(b) The graph of a particular solution to a differential equation passes through the point $(0,1)$.

On the slope field provided on the separate sheet, sketch the graph of this particular solution.

1

(c) A hard-boiled egg at $98°C$ is put in a room at $18°C$. After 5 minutes, the egg's temperature is $58°C$. The rate of cooling of the egg is proportional to the excess of the temperature T of the egg over the temperature S of the room, i.e. $\frac{dT}{dt} = k\left(T - S\right)$ where k is a constant and t is time in minutes.

(i) Show that $k = \frac{-\ln 2}{5}$. **2**

(ii) When will the temperature of the egg drop to $20°C$? Correct answer to the nearest minute. **2**

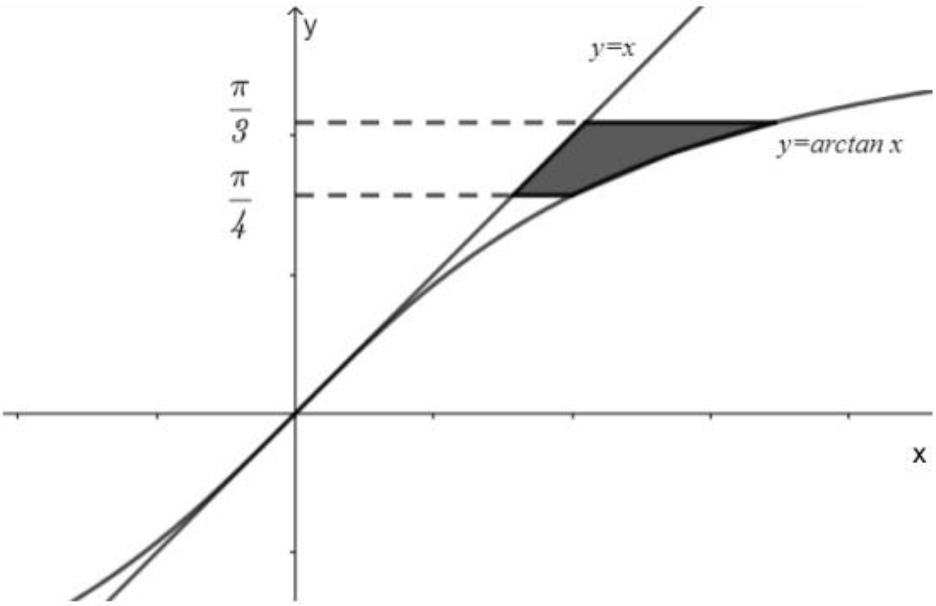
(d) (i) Show that $\frac{\sin\theta + \sin 3\theta + \sin 5\theta}{\cos\theta + \cos 3\theta + \cos 5\theta} = \tan 3\theta$ **2**

(ii) Hence solve $\frac{\sin\theta + \sin 3\theta + \sin 5\theta}{\cos\theta + \cos 3\theta + \cos 5\theta} = 1$, where $0 \le \theta \le 2\pi$. **2**

End of Question 12

Question 13 (15 Marks) Start a new writing booklet.

(a) Find the volume formed when the region between the curve $y = \tan^{-1} x$ and the lines $y = x$, $y = \frac{\pi}{3}$ and $y = \frac{\pi}{4}$, is rotated around the y axis, correct to 2 decimal places. **4**



Not To Scale

(b) Prove by mathematical induction that, for all integers $n \geq 1$, **3**

$$\frac{2}{1\times 3}+\frac{2}{2\times 4}+\frac{2}{3\times 5}+...+\frac{2}{n(n+2)}=\frac{3}{2}-\frac{2n+3}{(n+1)(n+2)}$$

Question 13 continues on the next page

(c) At a carnival, an air cannon fires a cylinder containing a t-shirt at time $t = 0$ seconds from an origin O at ground level across a level field. The position vector is

$$\underset{\sim}{r}(t) = 15\sqrt{3}t\underset{\sim}{i} + (15t - 4.9t^2)\underset{\sim}{j},$$

Where $\underset{\sim}{i}$ is a unit vector in the forward direction, $\underset{\sim}{j}$ is a unit vector vertically up and displacement components are measured in metres.

(i) Find the initial velocity of the t-shirt and the initial angle, in degrees, of its trajectory to the horizontal. **2**

(ii) Find the maximum height reached by the t-shirt, giving your answer in metres to two decimal places. **2**

(iii) Find the time of the flight of the t-shirt. Give your answer in seconds, correct to three decimal places. **1**

(iv) Find the range of the t-shirt in metres, correct to 1 decimal place. **1**

(v) A person in the crowd, more than 40 m from *O*, catches the t-shirt at a height of 2 m above the ground. How far horizontally from O, is this person when the t-shirt is caught? Give your answer to one decimal place **2**

End of Question 13

Question 14 (15 marks) Start a new writing booklet.

(a) The roots of $x^3 + 3x^2 - 4 = 0$ are α, β and γ. **2**

What is the value of $\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\alpha\gamma}$?

(b) The population $P(t)$ of bacteria in a petri dish is modelled by the logistic differential equation $\frac{dP}{dt} = \frac{P}{6}\left(1 - \frac{P}{8000}\right)$ where $P(0) = P_0$ and t is the time in hours.

(i) If the initial population P_0 is 1000 bacteria, show that $P(t) = \frac{8000}{1 + 7e^{\frac{-t}{6}}}$. **3**

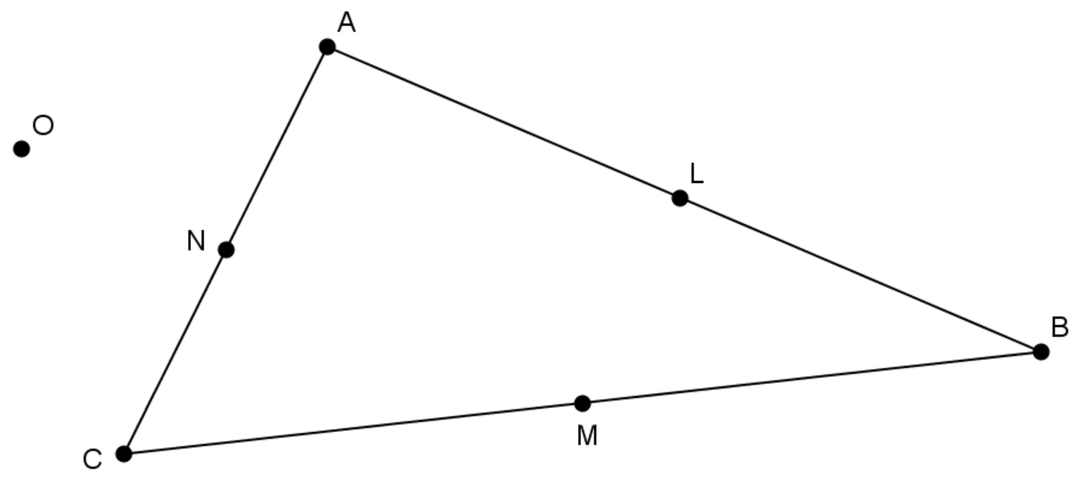
(You may use the fact that $\frac{Q}{P(Q-P)} = \frac{1}{P} + \frac{1}{Q-P}$)

(ii) If instead the initial population P_0 is 12 000 bacteria, describe what would have happened to the population as t increases. **1**

Question 14 continues on the next page

Question 14 continued

(c) The diagram shows O as the origin and a triangle with vertices A, B and C. Let the vectors $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{c}$. L, M and N are the midpoints of $\overrightarrow{AB}$, $\overrightarrow{CB}$ and $\overrightarrow{AC}$ respectively.



(i) Show that quadrilateral $LMNA$ is a parallelogram. **3**

(ii) Deduce that a line through the midpoints of two sides of triangle is half the length of the third side. **1**

(d) The polynomial $g(x) = 4x^3 - 3x + 1$ passes through the point $(1,\ 2)$.

(i) Show that $g(x)$ has a root of multiplicity of 2. **2**

(ii) Find the gradient of the tangent to $f(x) = \dfrac{g^{-1}(x)}{x}$ at the point where $x = 2$. **3**

End of paper