2025 HSC Or Otherwise Questions

by Ty Webb

Since 1917 it has become fashionable to put "or otherwise" in the final school exams in New South Wales for some questions. [1]

The trend evidently hasn't gone away as we see several of them in the 2025 exams [2], [3], [4]:

Advanced: Q29b

Extension 1: Q 12eii, Q13eii

Extension 2 Q13cii, Q14aii, Q15ciii, Q16a, Q16ciii

In this document we procure solutions to these questions with or otherwise methods. These may or may not be within the scope of the syllabus. Some are obviously not, some obviously are and others may be considered borderline and there will be disagreements about those and so I'm not going to delineate which are and which are not.

Advanced Q29b

By the Law of Sines for Tetrahedra,

$\sin\angle OFY \cdot \sin\angle OYT \cdot \sin\angle OTF = \sin\angle OFT \cdot \sin\angle OTY \cdot \sin\angle OYF$ whereupon

$\angle OFY = \sin^{-1}\frac{\sin 45^\circ \cdot \sin 30^\circ \cdot \sin 45^\circ}{\sin 60^\circ \cdot \sin 45^\circ} = \sin^{-1}\frac{1}{\sqrt{6}}$ and

$h = OT = OF = \frac{4\sin 45^\circ}{\sin(180^\circ - \sin^{-1}\frac{1}{\sqrt{6}} - 45^\circ)} = \sqrt{30} - \sqrt{6} \approx 3.03$ km.

Extension 1 Q12eii

For $0 \le x \le 2\pi, \sqrt{3}\sin x = \cos x + 1$ so

$3\sin^2 x = 3 - 3\cos^2 x = (\cos x + 1)^2 = \cos^2 x + 2\cos x + 1$

$4\cos^2 x + 2\cos x - 2 = 2(2\cos^2 x + \cos x - 1) = 2(\cos x + 1)(2\cos x - 1) = 0$

$\cos x = -1, \frac{1}{2}$ so $x = \pi, \frac{\pi}{3}, \frac{5\pi}{3}$

$\sqrt{3}\sin\pi = 0 = \cos\pi + 1$ and $\sqrt{3}\sin\frac{\pi}{3} = \frac{3}{2} = \cos\frac{\pi}{3} + 1$

but $\sqrt{3}\sin\frac{5\pi}{3} = -\frac{3}{2} \neq \frac{3}{2} = \cos\frac{5\pi}{3} + 1$ so final answer is $x = \pi, \frac{\pi}{3}$

Extension 1 Q13eii

$\sum\limits_{i=0}^{m} \binom{n+i}{n} = \binom{n+m+1}{n+1}$

Proof. $\sum\limits_{i=0}^{0} \binom{n+i}{n} = \binom{n}{n} = 1 = \binom{n+1}{n+1}$ so it is true for $m = 0$

If it is true for $m = k$ then $\sum\limits_{i=0}^{k} \binom{n+i}{n} = \binom{n+k+1}{n+1}$ and

$$\begin{aligned}\sum_{i=0}^{k+1} \binom{n+i}{n} &= \binom{n+k+1}{n} + \sum_{i=0}^{k} \binom{n+i}{n} \\ &= \binom{n+k+1}{n} + \binom{n+k+1}{n+1} \\ &= \binom{n+k+2}{n+1} \\ &= \binom{n+(k+1)+1}{n+1}\end{aligned}$$

so it is true for $m = k + 1$ so by the principle of mathematical induction it is true for all integers $m \geq 0$ $\square$

Now with $n = 2000, m = 50$ we have $\sum\limits_{i=0}^{50} \binom{n+i}{n} = \binom{2051}{2001}$

Extension 2 Q13cii

$$\begin{aligned} 3 &< 4 \\ 4n^2 + 8n + 3 &< 4n^2 + 8n + 4 \\ (2n+1)(2n+3) &< (2n+2)^2 \\ \sqrt{2n+1}\sqrt{2n+3} &< 2n+2 \\ \tfrac{\sqrt{2n+1}}{2n+2} &< \tfrac{1}{\sqrt{2n+3}} \\ \tfrac{2n+1}{2n+2} &< \tfrac{\sqrt{2n+1}}{\sqrt{2n+3}} \end{aligned}$$

Extension 2 Q14aii

We use the hypergeometric function: $I_n = \frac{{}_2F_1(1,\frac{2n+1}{2};\frac{2n+3}{2};-1)}{2n+1}$ so $I_2 = \frac{{}_2F_1(1,\frac{5}{2};\frac{7}{2};-1)}{5} = \frac{3\pi-8}{12}$

or digamma function: $I_n = \frac{1}{4}(\psi(\frac{2n+3}{4}) - \psi(\frac{2n+1}{4}))$ so $I_2 = \frac{1}{4}(\psi(\frac{7}{4}) - \psi(\frac{5}{4})) = \frac{3\pi-8}{12}$

Extension 2 Q15ciii

$z_0 = i\cot\frac{\pi}{12} = i\cot(\frac{\pi}{3} - \frac{\pi}{4}) = i \cdot \frac{1+\tan\frac{\pi}{3}\tan\frac{\pi}{4}}{\tan\frac{\pi}{3} - \tan\frac{\pi}{4}} = i \cdot \frac{1+\sqrt{3}}{\sqrt{3}-1} \cdot \frac{\sqrt{3}+1}{\sqrt{3}+1} = i \cdot \frac{4+2\sqrt{3}}{3-1} = (2+\sqrt{3})i$

$z_1 = i\cot\frac{3\pi}{12} = i$

$z_2 = i\cot\frac{5\pi}{12} = i\cot(\frac{5\pi}{3} - \frac{5\pi}{4}) = i \cdot \frac{1+\tan\frac{5\pi}{3}\tan\frac{5\pi}{4}}{\tan\frac{5\pi}{3} - \tan\frac{5\pi}{4}} = i \cdot \frac{1-\sqrt{3}}{-\sqrt{3}-1} \cdot \frac{-\sqrt{3}+1}{-\sqrt{3}+1} = i \cdot \frac{4-2\sqrt{3}}{3-1} = (2-\sqrt{3})i$

$z_3 = i\cot\frac{7\pi}{12} = i\cot(\pi - \frac{5\pi}{12}) = -i\cot\frac{5\pi}{12} = -(2-\sqrt{3})i$

$z_4 = i\cot\frac{9\pi}{12} = -i$

$z_5 = i\cot\frac{11\pi}{12} = i\cot(\pi - \frac{\pi}{12}) = -i\cot\frac{\pi}{12} = -(2+\sqrt{3})i$

are the roots of $\prod_{k=0}^{5}(z - z_k) = 0$ and pairing roots with same modulus and opposite sign

$$\begin{aligned}\prod_{k=0}^{5}(z-z_k) &= (z-i)(z+i)(z-(2+\sqrt{3})i)(z+(2+\sqrt{3})i)(z-(2-\sqrt{3})i)(z+(2-\sqrt{3})i)\\ &= (z^2+1)(z^2+7+4\sqrt{3})(z^2+7-4\sqrt{3})\\ &= (z^2+1)(z^4+14z^2+1)\\ &= z^6+15z^4+15z^2+1\\ &= \tfrac{1}{2}((z-1)^6+(z+1)^6)\\ &= 0\end{aligned}$$

so the roots of $\left(\frac{z-1}{z+1}\right)^6 = -1$ are

$i\cot(\frac{\pi}{12}), i\cot(\frac{3\pi}{12}), i\cot(\frac{5\pi}{12}), i\cot(\frac{7\pi}{12}), i\cot(\frac{9\pi}{12}), i\cot(\frac{11\pi}{12})$

Extension 2 Q16a

This uses the Cauchy bound

Proposition. $\forall z \in \mathbb{C}\backslash\{0\}\ \forall n \in \mathbb{Z}^+ \forall k \in \{0,1,2,\ldots,n\} \forall a_k \in \mathbb{C} : a_0 \neq 0 \wedge a_n \neq 0$

$$\text{if } P(z) = \sum_{k=0}^{n} a_k z^k \text{ and } P(z) = 0 \text{ then } \frac{1}{1+\max\limits_{1\le k\le n}\left|\frac{a_k}{a_0}\right|} < |z| < 1+\max_{0\le k\le n-1}\left|\frac{a_k}{a_n}\right|$$

Proof. If $|z| > 1$,

$$|a_n||z^n| = \left|\sum_{k=0}^{n-1} a_k z^k\right| \le \sum_{k=0}^{n-1}|a_k z^k| \le \max_{0\le k\le n-1}|a_k| \sum_{k=0}^{n-1}|z^k| = \frac{|z|^n-1}{|z|-1}\max_{0\le k\le n-1}|a_k| < \frac{|z|^n}{|z|-1}\max_{0\le k\le n-1}|a_k|$$

so $|a_n|(|z|-1) < \max\limits_{0\le k\le n-1}|a_k|$ but if $0 < |z| \le 1$ this inequality still holds since then $|a_n|(|z|-1) \le 0 < \max\limits_{0\le k\le n-1}|a_k|$ so we now have that

$$\text{for } |z| > 0, |z| < 1 + \max_{0\le k\le n-1}\left|\frac{a_k}{a_n}\right| \ldots (*)$$

If $w = \frac{1}{z}$ and $Q(w) = w^n P(\frac{1}{w}) = \sum_{k=0}^{n} a_{n-k} w^k$ then from (*) if $Q(w) = 0$

then $|w| < 1 + \max\limits_{1 \le k \le n} \left|\frac{a_k}{a_0}\right|$ and $|z| > \frac{1}{1+\max\limits_{1 \le k \le n}\left|\frac{a_k}{a_0}\right|}$ so now

$\frac{1}{1+\max\limits_{1 \le k \le n}\left|\frac{a_k}{a_0}\right|} < |z| < 1 + \max\limits_{0 \le k \le n-1}\left|\frac{a_k}{a_n}\right| \qquad \square$

With $a_0 = -1$ and for $k > 0, a_k = \cos k\theta$ since $0 \le |\cos k\theta| \le 1$ if $P(z) = -1 + \sum\limits_{k=1}^{n} z^k \cos k\theta = 0$ then $\frac{1}{2} < |z| < 1 + |\sec n\theta|$

This or otherwise method has an advantage in that it also establishes an upper bound.

Extension 2 Q16ciii

If P is a point not on ℓ and A is the point on $\ell : \underset{\sim}{a} + \lambda \underset{\sim}{d}$ with position vector $\underset{\sim}{a}$ then the minimum distance from P to ℓ is $\frac{|\overrightarrow{PA} \times \underset{\sim}{d}|}{|\underset{\sim}{d}|}$ so where O is the origin, the minimum distance from ℓ to O is $\frac{|\overrightarrow{OA} \times \underset{\sim}{d}|}{|\underset{\sim}{d}|} = |\underset{\sim}{a} \times \underset{\sim}{d}|$ since $|\underset{\sim}{d}| = 1$ and so the minimum distance from ℓ to the sphere is

$$\begin{cases} 0 & \text{if } |\underset{\sim}{a} \times \underset{\sim}{d}| \le 1 \\ |\underset{\sim}{a} \times \underset{\sim}{d}| - 1 & \text{if } |\underset{\sim}{a} \times \underset{\sim}{d}| > 1 \end{cases}$$

Note this can also be expressed with dot product as follows.

For vectors $\underset{\sim}{u}, \underset{\sim}{v}$, $|\underset{\sim}{u} \times \underset{\sim}{v}|^2 = |\underset{\sim}{u}|^2|\underset{\sim}{v}|^2 - (\underset{\sim}{u} \cdot \underset{\sim}{v})^2$ so

$|\underset{\sim}{a} \times \underset{\sim}{d}|^2 = |\underset{\sim}{a}|^2|\underset{\sim}{d}|^2 - (\underset{\sim}{a} \cdot \underset{\sim}{d})^2 = |\underset{\sim}{a}|^2 - (\underset{\sim}{a} \cdot \underset{\sim}{d})^2$ (since $|\underset{\sim}{d}| = 1$) and answer is

$$\begin{cases} 0 & \text{if } |\underset{\sim}{a}|^2 - (\underset{\sim}{a} \cdot \underset{\sim}{d})^2 \le 1 \\ \sqrt{|\underset{\sim}{a}|^2 - (\underset{\sim}{a} \cdot \underset{\sim}{d})^2} - 1 & \text{if } |\underset{\sim}{a}|^2 - (\underset{\sim}{a} \cdot \underset{\sim}{d})^2 > 1 \end{cases}$$

References.

[1] NSW Department of Education, 1917 Mathematics Honours I Leaving Certificate Examination

[2] NESA, 2025 Mathematics Advanced HSC Examination

[3] NESA, 2025 Mathematics Extension 1 HSC Examination

[4] NESA, 2025 Mathematics Extension 2 HSC Examination