Mathematics Extension 2 HSC 2026 Sixteen Methods for Question 8

by Ty Webb

8. Method 1. Half base times height

$\frac{1}{2}|2z-(2z-iz)||-iz-(2z-iz)| = |z|^2 \therefore$ B

8. Method 2. Determinant

Area of triangle with vertices z_1, z_2, z_3 is $\frac{1}{4}\left|\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}\right|$ and so area is

$\frac{1}{4}\left|\begin{vmatrix} 2z & 2\bar{z} & 1 \\ -iz & \overline{-iz} & 1 \\ 2z-iz & \overline{2z-iz} & 1 \end{vmatrix}\right|$

$= \frac{1}{4}|2z(\overline{-iz}(1)-1(\overline{2z-iz}))-2\bar{z}(-iz(1)-1(2z-iz))+1(-iz(\overline{2z-iz})-\overline{-iz}(2z-iz))|$

$= \frac{|z\bar{z}|}{4}|2(i-(2+i))-2(-i-(2-i))-i(2+i)-i(2-i)|$

$= \frac{|z|^2}{4}|-4+4-4i|$

$= |z|^2$

$\therefore$ B.

8. Method 3. Shoelace formula with determinants

Area of convex polygon with vertices $z_1, z_2, \ldots, z_n$ is $\frac{1}{4}\left|\sum_{k=1}^{n}\begin{vmatrix} \bar{z}_k & z_k \\ \bar{z}_{k+1} & z_{k+1} \end{vmatrix}\right|$ where $z_{n+1} = z_1$ and so area of triangle is $\frac{1}{4}\left|\begin{vmatrix} \bar{z}_1 & z_1 \\ \bar{z}_2 & z_2 \end{vmatrix}+\begin{vmatrix} \bar{z}_2 & z_2 \\ \bar{z}_3 & z_3 \end{vmatrix}+\begin{vmatrix} \bar{z}_3 & z_3 \\ \bar{z}_1 & z_1 \end{vmatrix}\right|$ and so the area is

$\frac{1}{4}\left|\begin{vmatrix} 2\bar{z} & 2z \\ \overline{-iz} & -iz \end{vmatrix}+\begin{vmatrix} \overline{-iz} & -iz \\ \overline{2z-iz} & 2z-iz \end{vmatrix}+\begin{vmatrix} \overline{2z-iz} & 2z-iz \\ 2\bar{z} & 2z \end{vmatrix}\right|$

$= \frac{1}{4}|2\bar{z}(-iz)-2z(\overline{-iz})+\overline{-iz}(2z-iz)-(-iz)\overline{2z-iz}+\overline{2z-iz}(2z)-(2z-iz)(2\bar{z})|$

$= \frac{1}{4}|z\bar{z}(-2i-2i+2i+1+2i-1+4+2i-4+2i)|$

$= \frac{4|z|^2}{4}$

$= |z|^2$

$\therefore$ B.

8. Method 4. Shoelace formula without determinants

Area of convex polygon with vertices $z_1, z_2, \ldots, z_n$ is $\frac{1}{2}\left|\mathrm{Im}\left(\sum_{k=1}^{n}\bar{z}_k z_{k+1}\right)\right|$ where $z_{n+1} = z_1$ so area of triangle is $\frac{1}{2}|\mathrm{Im}(\bar{z}_1 z_2+\bar{z}_2 z_3+\bar{z}_3 z_1)|$ and so the area is

$\frac{1}{2}|\mathrm{Im}(2\bar{z}(-iz)+\overline{-iz}(2z-iz)+\overline{2z-iz}(2z))|$

$= \frac{1}{2}|\mathrm{Im}(z\bar{z}(-2i+2i+1+4+2i))|$

$= \frac{1}{2}|\mathrm{Im}(|z|^2(5+2i))|$

$= \frac{2|z|^2}{2}$

$= |z|^2$

$\therefore$ B.

8. Method 5. Simplified shoelace formula without determinants

Area of triangle with vertices z_1, z_2, z_3 is $\frac{1}{2}|\text{Im}((z_2 - z_1)(\overline{z_3 - z_1}))|$ hence area of triangle is

$\frac{1}{2}|\text{Im}((-iz - 2z)(\overline{2z - iz - 2z}))| = \frac{1}{2}|\text{Im}(z\bar{z}(-i - 2)(i))| = \frac{|z|^2}{2}|\text{Im}(1 - 2i)| = |z|^2 \therefore$ B.

8. Method 6. SSS (Heron's formula)

$a = |-iz - 2z| = |-i - 2||z| = \sqrt{1^2 + 2^2}|z| = \sqrt{5}|z|,$
$b = |2z - iz - 2z| = |-i||z| = |z|,$
$c = |2z - iz - (-iz)| = |2||z| = 2|z|,$
$s = \frac{a+b+c}{2} = \frac{(3+\sqrt{5})|z|}{2}, s - a = \frac{(3-\sqrt{5})|z|}{2}, s - b = \frac{(1+\sqrt{5})|z|}{2}, s - c = \frac{(-1+\sqrt{5})|z|}{2}$

$$\begin{aligned}\Rightarrow \text{Area} &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \frac{|z|^2}{4}\sqrt{(3+\sqrt{5})(3-\sqrt{5})(1+\sqrt{5})(-1+\sqrt{5})} \\ &= \frac{|z|^2}{4}\sqrt{(9-5)(-1+5)} \\ &= |z|^2\end{aligned}$$

$\therefore$ B

The following methods may use sides found in Method 6 giving angles

$\hat{A} = \cos^{-1}\frac{b^2+c^2-a^2}{2bc} = \cos^{-1}\frac{1+4-5}{2\times1\times2} = \frac{\pi}{2},$

$\hat{B} = \cos^{-1}\frac{a^2+c^2-b^2}{2ab} = \cos^{-1}\frac{5+4-1}{2\times\sqrt{5}\times2} = \cos^{-1}\frac{2}{\sqrt{5}}$

$\hat{C} = \cos^{-1}\frac{a^2+b^2-c^2}{2ab} = \cos^{-1}\frac{5+1-4}{2\times\sqrt{5}\times1} = \cos^{-1}\frac{1}{\sqrt{5}}$

and altitudes

$h_a = b\sin C = |z|\sin\cos^{-1}\frac{1}{\sqrt{5}} = \frac{2|z|}{\sqrt{5}}$

$h_b = a\sin C = \sqrt{5}|z|\sin\cos^{-1}\frac{1}{\sqrt{5}} = 2|z|$

$h_c = a\sin B = \sqrt{5}|z|\sin\cos^{-1}\frac{2}{\sqrt{5}} = |z|$

8. Method 7. SAS ($\frac{1}{2}ab\sin C$)

$\frac{1}{2}\sqrt{5}|z||z|\sin\cos^{-1}\frac{1}{\sqrt{5}} = |z|^2 \therefore$ B

8. Method 8. AAS ($\frac{b^2\sin A\sin(A+B)}{2\sin B}$)

$\frac{|z|^2\sin\frac{\pi}{2}\sin(\frac{\pi}{2}+\cos^{-1}\frac{2}{\sqrt{5}})}{2\sin\cos^{-1}\frac{2}{\sqrt{5}}} = |z|^2 \therefore$ B

8. Method 9. ASA ($\frac{a^2 \sin B \sin C}{2 \sin(B+C)}$)

$\frac{5|z|^2 \sin\cos^{-1}\frac{2}{\sqrt{5}} \sin\cos^{-1}\frac{1}{\sqrt{5}}}{2\sin(\cos^{-1}\frac{2}{\sqrt{5}}+\cos^{-1}\frac{1}{\sqrt{5}})} = |z|^2 \therefore$ B

8. Method 10. Tangent ($\frac{1}{4}(a^2+b^2-c^2)\tan C$)

$\frac{1}{4}(5|z|^2+|z|^2-4|z|^2)\tan\cos^{-1}\frac{1}{\sqrt{5}} = |z|^2 \therefore$ B

8. Method 11. 3 sides and 3 altitudes ($\frac{1}{2}\sqrt[3]{abch_a h_b h_c}$)

$\frac{1}{2}\sqrt[3]{\sqrt{5}|z| \times |z| \times 2|z| \times \frac{2|z|}{\sqrt{5}} \times 2|z| \times |z|} = |z|^2 \therefore$ B

8. Method 12. 2 sides and 2 altitudes ($\frac{1}{2}\sqrt{abh_a h_b}$)

$\frac{1}{2}\sqrt{\sqrt{5}|z| \times |z| \times \frac{2|z|}{\sqrt{5}} \times 2|z|} = |z|^2 \therefore$ B

8. Method 13. 2 sides and 1 altitude ($\frac{1}{2}h_c(\sqrt{a^2-h_c^2}+\sqrt{b^2-h_c^2})$)

$\frac{1}{2}|z|(\sqrt{5|z|^2-|z|^2}+\sqrt{|z|^2-|z|^2}) = |z|^2 \therefore$ B

8. Method 14. 2 altitudes and 1 angle ($\frac{h_a h_b}{2\sin C}$)

$\frac{\frac{2|z|}{\sqrt{5}} \times 2|z|}{2\sin\cos^{-1}\frac{1}{\sqrt{5}}} = |z|^2 \therefore$ B

8. Method 15. 1 altitude and 2 angles ($\frac{1}{2}h_a^2(\cot B+\cot C)$)

$\frac{1}{2} \times \frac{4|z|^2}{5}(\cot\cos^{-1}\frac{2}{\sqrt{5}}+\cot\cos^{-1}\frac{1}{\sqrt{5}}) = |z|^2 \therefore$ B

8. Method 16. Cotangents ($\frac{1}{4}(a^2\cot A+b^2\cot B+c^2\cot C)$)

$\frac{1}{4}(5|z|^2\cot\frac{\pi}{2}+|z|^2\cot\cos^{-1}\frac{2}{\sqrt{5}}+4|z|^2\cot\cos^{-1}\frac{1}{\sqrt{5}}) = |z|^2 \therefore$ B

Note: I based the last 6 of these on the January 7, 1885 presentation to the Mathematical section of the Philosophical Society of Washington which was subsequently published in 2 articles

Baker, Marcus, "A collection of formulae for the area of a plane triangle," Annals of Mathematics, part 1 in vol. 1(6), January 1885, 134–138; part 2 in vol. 2(1), September 1885, 11–18

This is 110 formulae for area of a triangle so that actually gives 110 methods for Question 8!

Some might say this is the sort of thing we might expect AI to come up with. But this was 141 years ago!

A combined pdf of the 2 articles with all 110 formulae can be found at

https://4unitmaths.com/baker-110.pdf

Just be warned that some have typos and need to be corrected. So if one were to up the ante with this and actually do a 110 methods version of this article one should use corrected versions of some of Baker's formulae.