Blacktown Boys' High School

2023

HSC Trial Examination

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided for this paper
- All diagrams are not drawn to scale
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks (pages 3 – 6)
- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 7 – 12)
- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

Assessor: X. Chirgwin

Student Name: ______________________________

Teacher Name: ______________________________

Students are advised that this is a trial examination only and cannot in any way guarantee the content or format of the 2023 Higher School Certificate Examination.

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Section I
10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1–10.

Q1 Let $\underset{\sim}{a} = 3\underset{\sim}{i} - 5\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} - 4\underset{\sim}{k}$. Which of the following is equal to $\underset{\sim}{b} - 2\underset{\sim}{a}$?

A. $\underset{\sim}{i} - 5\underset{\sim}{j} + 7\underset{\sim}{k}$

B. $-5\underset{\sim}{i} + 5\underset{\sim}{j} - 3\underset{\sim}{k}$

C. $-5\underset{\sim}{i} + 10\underset{\sim}{j} - 2\underset{\sim}{k}$

D. $-5\underset{\sim}{i} + 6\underset{\sim}{j} + 2\underset{\sim}{k}$

Q2 If $z = 2e^{-\frac{i\pi}{3}}$, then z^4 is equal to

A. $-16\left(\cos\left(-\frac{4\pi}{3}\right) + i\sin\left(-\frac{4\pi}{3}\right)\right)$

B. $-16\left(\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right)$

C. $16\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$

D. $16\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$

Q3 The negation of the statement "I listen to music and I play games." is:

A. I don't listen to music or I don't play games.

B. I don't listen to music and I don't play games.

C. I don't listen to music or I play games.

D. I don't listen to music and I play games.

Q4 The polynomial equation $P(z) = 0$ has real coefficients. Three of the roots of this equation are $z = -3 + 2i$, $z = i$ and $z = 0$.

The minimum degree of $P(z)$ is

A. 3

B. 4

C. 5

D. 6

Q5 A particle is moving in simple harmonic motion and its displacement, x units, at time t seconds is given by the equation $x = A\cos(nt) + 3$. The period of the motion is 6π seconds and the particle is initially at rest, 13 units to the right of the origin. Find the values of A and n.

A. $A = 10, n = \frac{1}{3}$

B. $A = 10, n = 3$

C. $A = 13, n = \frac{1}{3}$

D. $A = 13, n = 3$

Q6 Find $\int \frac{5x+1}{x^2+25}\,dx$

A. $\frac{5}{2}\log_e(x^2+25) + \tan^{-1}\left(\frac{x}{5}\right) + C$

B. $\frac{5}{2}\log_e(x^2+25) + \frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right) + C$

C. $5\log_e(x^2+25) + \tan^{-1}\left(\frac{x}{5}\right) + C$

D. $5\log_e(x^2+25) + \frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right) + C$

Q7 The complex numbers z, iz and $z + iz$, where z is a complex number, are plotted in the Argand plane, forming the vertices of a triangle. The area of this triangle is

A. $|z| + |z|^2$

B. $|z|^2$

C. $\dfrac{\sqrt{2}|z|^2}{2}$

D. $\dfrac{|z|^2}{2}$

Q8 Let $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j} + 6\underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} - 4\underset{\sim}{j} + 8\underset{\sim}{k}$, where the acute angle between the vectors is θ.

The value of $\sin(2\theta)$ is

A. $\dfrac{175\sqrt{101}}{63^2}$

B. $\dfrac{380\sqrt{101}}{63^2}$

C. $\dfrac{380\sqrt{101}}{63}$

D. $\dfrac{320\sqrt{5}}{63}$

Q9 Given that $x, y \epsilon \mathbb{Z}$, where $x, y \geq 0$, which of the following is a FALSE statement?

A. $\exists x(\forall y : y = 1 - 3x)$

B. $\exists x(\exists y : y = 5 + 2x)$

C. $\forall x(\exists y : y = x^2)$

D. $\forall x(\exists y : y - x = 0)$

Q10 All of the integrals below are of the form $\displaystyle\int_{-1}^{1} f(x)\,dx$.

Which of the integrals can be written as $\displaystyle 2\int_{0}^{1} f(x)\,dx$?

A. $\displaystyle\int_{-1}^{1} e^{2x} \tan^{-1}\left(\frac{x}{2}\right) dx$

B. $\displaystyle\int_{-1}^{1} \sqrt{2\tan^2 x + x^3}\, dx$

C. $\displaystyle\int_{-1}^{1} \frac{x^3 \sin^2 x}{x^2 + 2} dx$

D. $\displaystyle\int_{-1}^{1} x^3 \sin^{-1}\left(\frac{x}{2}\right) dx$

End of Section I

Section II

90 Marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question on a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Given the two complex numbers $z = \sqrt{6} - \sqrt{2}i$ and $w = 2e^{\frac{i\pi}{4}}$,

(i) Express z in the form $re^{i\theta}$. **1**

(ii) Find wz^6 in $x + iy$ form. **2**

(b) Evaluate

(i) $\displaystyle\int_0^1 \frac{x^2}{x+2}\,dx$ **2**

(ii) $\displaystyle\int_0^3 \sqrt{\frac{6-x}{6+x}}\,dx$ **3**

(c) Prove the following statement by contradiction: **2**

"If n is odd, then $5n^2 - 8n$ is odd."

(d) Given that $\underset{\sim}{a} = -3\underset{\sim}{i} + 2\underset{\sim}{j} + 17\underset{\sim}{k}$ and $\underset{\sim}{b} = -m\underset{\sim}{i} + 6\underset{\sim}{j} + 14\underset{\sim}{k}$, where m is a positive real constant, the vector $\underset{\sim}{a} - \underset{\sim}{b}$ is perpendicular to vector $\underset{\sim}{b}$.

(i) Find the value of m. **3**

(ii) Find the vector projection of the vector $\underset{\sim}{a} - \underset{\sim}{b}$ onto vector $\underset{\sim}{c}$, where $\underset{\sim}{c} = 2\underset{\sim}{i} - 5\underset{\sim}{j} - 4\underset{\sim}{k}$. **2**

End of Questions 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) A line in the complex plane is given by $|z - 3| = |z + 1 - 4i|, z \in \mathbb{C}$.

(i) Find the Cartesian equation of the line. **2**

(ii) Find the points of intersection of the line with the circle $|z - 3| = 4$. **2**

(iii) Sketch both the line and the circle on the Argand diagram, showing clearly all key features and the points of intersection. **2**

(iv) The line cuts the circle into two segments. Find the exact area of the minor segment. **1**

(b) (i) Express $\dfrac{11x + 10}{(x^2+4)(x-2)}$ as a sum of partial fractions over $\mathbb{R}$. **2**

(ii) Hence evaluate $\displaystyle\int_3^4 \frac{11x+10}{(x^2+4)(x-2)}\,dx$. **2**

(c) Find

(i) $\displaystyle\int \frac{2}{\sqrt{8x - 12 - x^2}}\,dx$ **2**

(ii) $\displaystyle\int \sqrt{2x}\ln x\,dx$ **2**

End of Questions 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the polynomial $P(x) = x^4 - 8x^3 + 42x^2 - 104x + 1769$.

(i) If $P(x)$ has roots $a + bi, -3a + bi$ (where a, b are real), find the values of a and b. **2**

(ii) Hence express $P(x)$ as the product of two real quadratic factors. **2**

(b) Relative to a fixed origin O, the points A, B and C have respective position vectors $\underset{\sim}{i} + 10\underset{\sim}{k}$, $4\underset{\sim}{i} + 3\underset{\sim}{j} + 7\underset{\sim}{k}$ and $8\underset{\sim}{i} + 7\underset{\sim}{j} + 3\underset{\sim}{k}$.

(i) Show that A, B and C are collinear, and find the ratio $AB: BC$. **2**

(ii) Find a vector equation for the straight line l that passes through A, B and C. **1**

(iii) Show that OB is perpendicular to l. **1**

(iv) Calculate the area of the triangle OAC. **2**

(c) A particle moves in a straight line and at time t seconds, its velocity, v metres per second, is related to its displacement, x metres, by

$$v^2 = 189 - 42x - 7x^2$$

(i) Show that the motion is simple harmonic in the form of $\ddot{x} = -n^2(x - c)$ **1**

(ii) Find the period of the motion. **1**

(iii) Find the amplitude of the motion. **2**

(iv) What is the maximum speed of this particle and what is the value of x when the particle is at the maximum speed? **1**

End of Questions 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Use the substitution $u = \frac{\pi}{2} - x$ to show that **2**

$$\int_0^{\frac{\pi}{2}} \frac{e^{\sin x}}{e^{\sin x} + e^{\cos x}} dx = \int_0^{\frac{\pi}{2}} \frac{e^{\cos x}}{e^{\sin x} + e^{\cos x}} dx$$

(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \frac{e^{\sin x}}{e^{\sin x} + e^{\cos x}} dx$ **1**

(b) Prove that for any integer $n > 1$, $\log_n(n - 1)$ is irrational. **2**

(c) Given that $z = e^{i\theta}$

(i) Show that $z^n + \frac{1}{z^n} = 2\cos(n\theta)$ **1**

(ii) Hence solve the follow equation and leave answers in $x + iy$ form where x and y are real. **4**

$$5z^4 - 11z^3 + 16z^2 - 11z + 5 = 0$$

(d) Let $I_n = \int \frac{dx}{(x^2 + 1)^n}$ where $n \geq 1$, is an integer.

(i) Show that, for $n \geq 2$, $I_n = \frac{1}{2(n-1)}\left[\frac{x}{(x^2 + 1)^{n-1}} + (2n - 3)I_{n-1}\right]$ **3**

(ii) Hence evaluate $\int_0^1 \frac{dx}{(x^2 + 1)^2}$ **2**

End of Questions 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{2}{4 + 5\cos x}\,dx$ **4**

(b) The velocity of a particle at position $x \geq 0$ is given by $v = 4e^{-2x} - 3$, and initially the particle is at $x = 0$.

(i) Find the initial acceleration of this particle. **2**

(ii) Find the displacement of the particle x in terms of time t. **3**

(iii) Find the velocity of the particle v in terms of time t. **1**

(c) If a, b, c and d are positive real numbers, prove that:

(i) $\frac{a+b}{2} \geq \sqrt{ab}$ **1**

(ii) $(a + b + c + d)^2 \geq 4(ac + bc + bd + ad)$ **2**

(iii) $(a + b + c + d)^2 \geq \frac{8}{3}(ab + ad + bc + cd + bd + ac)$ **2**

End of Questions 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) (i) By considering the interval $0 < x < 1$, show that **2**

$$\frac{1}{2}x(1-x)^3 < \frac{x(1-x)^3}{1+x} < x(1-x)^3$$

(ii) Deduce that **1**

$$\frac{1}{2}\int_0^1 x(1-x)^3 dx < \int_0^1 \frac{x(1-x)^3}{1+x} dx < \int_0^1 x(1-x)^3 dx$$

(iii) Given that $\int_0^1 \frac{x(1-x)^3}{1+x} dx = \frac{67}{12} - 8\ln 2$, deduce that **3**

$$\frac{83}{120} < \ln 2 < \frac{667}{960}$$

(b) A sequence $\{b_n\}$ is defined by $b_1 = 1$ and $b_{n+1} = b_n(b_n + 1)$, for all $n \geq 1$.

(i) Evaluate b_2, b_3, b_4. **1**

(ii) Use mathematical induction to prove that for each n, **3**

$$b_{n+1} = 1 + \sum_{r=1}^{n} (b_r)^2$$

(iii) Evaluate b_5 and express it as the sum of 5 positive squares. **1**

(iv) Show that $(2b_{n+1} + 1)^2 = (2b_n + 1)^2 + (2b_{n+1})^2$ **1**

(v) Hence deduce that $(2b_{n+1} + 1)^2 = (2b_1 + 1)^2 + \sum_{r=2}^{n+1} (2b_r)^2$ **2**

(vi) Hence prove that $3^2 + 4^2 + 12^2 + 84^2 + 3612^2 = 3613^2$ **1**

End of Paper

Student Name:____

Multiple Choice Answer Sheet

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word 'correct' and drawing an arrow as follows.

correct ↙

A ⊗ B ⊗ C ○ D ○

Start Here →

1. A ○ B ○ C ○ D ○
2. A ○ B ○ C ○ D ○
3. A ○ B ○ C ○ D ○
4. A ○ B ○ C ○ D ○
5. A ○ B ○ C ○ D ○
6. A ○ B ○ C ○ D ○
7. A ○ B ○ C ○ D ○
8. A ○ B ○ C ○ D ○
9. A ○ B ○ C ○ D ○
10. A ○ B ○ C ○ D ○

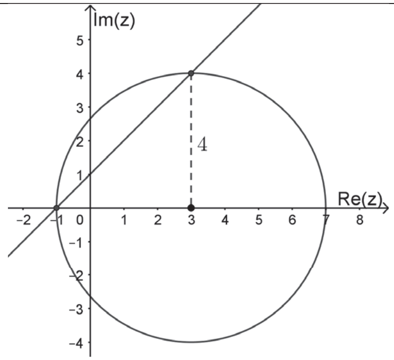
2023 Mathematics Extension 2 AT4 Trial Solutions

Section 1

Q1	**C** $\underset{\sim}{b} - 2\underset{\sim}{a}$ $= \underset{\sim}{i} - 4\underset{\sim}{k} - 2\left(3\underset{\sim}{i} - 5\underset{\sim}{j} - \underset{\sim}{k}\right)$ $= -5\underset{\sim}{i} + 10\underset{\sim}{j} - 2\underset{\sim}{k}$	1 Mark
Q2	**D** $z^4 = \left(2e^{-\frac{i\pi}{3}}\right)^4 = 2^4 e^{-\frac{i4\pi}{3}}$ $z^4 = 16\left(\cos\left(-\frac{4\pi}{3}\right) + i\sin\left(-\frac{4\pi}{3}\right)\right) = 16\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$	1 Mark
Q3	**A** The negation of "p and q" is "not p or not q" The negation statement is "I don't listen to music or I don't play games."	1 Mark
Q4	**C** Real polynomial, complex roots appear in conjugate pairs, so the minimum degree of $P(z)$ is 5.	1 Mark
Q5	**A** $T = \frac{2\pi}{n} = 6\pi$ $n = \frac{1}{3}$ $t = 0, x = 13$ $13 = A + 3$ $A = 10$	1 Mark
Q6	**B** $\int \frac{5x+1}{x^2+25}dx$ $= \frac{5}{2}\int \frac{2xdx}{x^2+25} + \int \frac{1}{x^2+25}dx$ $= \frac{5}{2}\log_e(x^2+25) + \frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right) + C$	1 Mark
Q7	**D** z, iz and $z + iz$ form a right-angled triangle. Area of this triangle is $\frac{1}{2} \times \lvert z\rvert \times \lvert iz\rvert = \frac{\lvert z\rvert^2}{2}$	1 Mark
Q8	**B** $\cos\theta = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{b}\rvert} = \frac{2 \times 1 + 3 \times -4 + 6 \times 8}{\sqrt{2^2+3^2+6^2} \times \sqrt{1^2+4^2+8^2}}$ $\cos\theta = \frac{38}{63}$ $\sin\theta = \frac{5\sqrt{101}}{63}$ $\sin(2\theta) = 2\sin\theta\cos\theta = 2 \times \frac{38}{63} \times \frac{5\sqrt{101}}{63} = \frac{380\sqrt{101}}{63^2}$	1 Mark

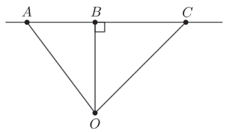
Q9	**A** Counter example $y = 2$ $2 = 1 - 3x$ $1 = -3x$ $x = -\frac{1}{3}$ Which is false.	1 Mark
Q10	**D** If $f(x)$ is even, $\int_{-1}^{1} f(x)\,dx = 2\int_{0}^{1} f(x)\,dx$ $f(x) = x^3 \sin^{-1}\left(\frac{x}{2}\right)$ $f(-x) = (-x)^3 \sin^{-1}\left(\frac{-x}{2}\right)$ $f(-x) = -x^3 \times -\sin^{-1}\left(\frac{x}{2}\right)$ $f(-x) = x^3 \sin^{-1}\left(\frac{x}{2}\right) = f(x)$	1 Mark

Section 2		
Q11ai	$z = \sqrt{6} - \sqrt{2}i$ $z = 2\sqrt{2}e^{-\frac{i\pi}{6}}$	1 Mark Correct solution
Q11aii	wz^6 $= 2e^{\frac{i\pi}{4}} \times \left(2\sqrt{2}e^{-\frac{i\pi}{6}}\right)^6$ $= 2e^{\frac{i\pi}{4}} \times 512e^{-i\pi}$ $= 1024e^{-\frac{3i\pi}{4}}$ $= 1024\left(\cos\left(-\frac{3i\pi}{4}\right) + i\sin\left(-\frac{3i\pi}{4}\right)\right)$ $= 1024\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$ $= -512\sqrt{2} - 512\sqrt{2}i$	2 Marks Correct solution 1 Mark Finds $z^6 = 64e^{-i\pi}$
Q11bi	$\int_0^1 \frac{x^2}{x+2}dx$ $= \int_0^1 \frac{x^2 - 4 + 4}{x+2}dx$ $= \int_0^1 \left(x - 2 + \frac{4}{x+2}\right)dx$ $= \left[\frac{x^2}{2} - 2x + 4\ln\lvert x+2\rvert\right]_0^1$ $= \left(\frac{1^2}{2} - 2 \times 1 + 4\ln\lvert 1+2\rvert\right) - (0 + 4\ln\lvert 0+2\rvert)$ $= -\frac{3}{2} + 4\ln\frac{3}{2}$	2 Marks Correct solution 1 Mark Correct primitive function
Q11bii	$\int_0^3 \sqrt{\frac{6-x}{6+x}}dx$ $= \int_0^3 \sqrt{\frac{6-x}{6+x}} \times \sqrt{\frac{6-x}{6-x}}dx$ $= \int_0^3 \frac{6-x}{\sqrt{36-x^2}}dx$ $= \int_0^3 \left(\frac{6}{\sqrt{36-x^2}} - \frac{x}{\sqrt{36-x^2}}\right)dx$ $= \left[6\sin^{-1}\frac{x}{6} + \sqrt{36-x^2}\right]_0^3$ $= \left(6\sin^{-1}\frac{3}{6} + \sqrt{36-3^2}\right) - \left(6\sin^{-1}0 + \sqrt{36-0^2}\right)$ $= \left(6 \times \frac{\pi}{6} + \sqrt{27}\right) - (0+6)$ $= \pi + 3\sqrt{3} - 6$	3 Marks Correct solution 2 Marks Correct primitive function 1 Mark Correct integration for either $\frac{6}{\sqrt{36-x^2}}$ or $\frac{x}{\sqrt{36-x^2}}$
Q11c	Prove by contradiction: "If n is odd, then $5n^2 - 8n$ is even." Then $n = 2k+1$ for some integer k. $5(2k+1)^2 - 8(2k+1)$ $= 5(4k^2 + 4k + 1) - 16k - 8$ $= 20k^2 + 20k + 5 - 16k - 8$ $= 20k^2 + 4k - 4 + 1$ $= 2(10k^2 + 2k - 2) + 1$ $= 2p + 1$, where $p = 10k^2 + 2k - 2$ Which is odd, therefore proved by contradiction.	2 Marks Correct solution 1 Mark Makes significant progress

Q11di	$\underset{\sim}{a}-\underset{\sim}{b}=\left(-3\underset{\sim}{i}+2\underset{\sim}{j}+17\underset{\sim}{k}\right)-\left(-m\underset{\sim}{i}+6\underset{\sim}{j}+14\underset{\sim}{k}\right)$ $\underset{\sim}{a}-\underset{\sim}{b}=(m-3)\underset{\sim}{i}-4\underset{\sim}{j}+3\underset{\sim}{k}$ $\left(\underset{\sim}{a}-\underset{\sim}{b}\right)\cdot\underset{\sim}{b}=0$ $(m-3)\times -m+(-4)\times 6+3\times 14=0$ $-m^2+3m+18=0$ $m^2-3m-18=0$ $(m-6)(m+3)=0$ $m=-3, m=6$ Since m is positive, then $m=6$.	3 Marks Correct solution 2 Marks Obtains $-m^2+2m=0$ 1 Mark Finds $\underset{\sim}{a}-\underset{\sim}{b}$
Q11dii	$\underset{\sim}{a}-\underset{\sim}{b}=3\underset{\sim}{i}-4\underset{\sim}{j}+3\underset{\sim}{k}$ $\frac{3\times 2+(-4)\times(-5)+3\times(-4)}{\sqrt{2^2+5^2+4^2}}\times\frac{1}{\sqrt{2^2+5^2+4^2}}\left(2\underset{\sim}{i}-5\underset{\sim}{j}-4\underset{\sim}{k}\right)$ $=\frac{14}{45}\left(2\underset{\sim}{i}-5\underset{\sim}{j}-4\underset{\sim}{k}\right)=\frac{28}{45}\underset{\sim}{i}-\frac{14}{9}\underset{\sim}{j}-\frac{56}{45}\underset{\sim}{k}$	2 Marks Correct solution 1 Mark Substitute into vector projection formula
Q12ai	$\lvert z-3\rvert=\lvert z+1-4i\rvert$ $(x-3)^2+y^2=(x+1)^2+(y-4)^2$ $x^2-6x+9+y^2=x^2+2x+1+y^2-8y+16$ $-8x+8y-8=0$ $x-y+1=0$ or $y=x+1$	2 Marks Correct solution 1 Mark Makes significant progress
Q12aii	$\lvert z-3\rvert=4$ $(x-3)^2+y^2=4^2 \ldots (1)$ $y=x+1 \ldots (2)$ Sub (2) into (1) $(x-3)^2+(x+1)^2=4^2$ $x^2-6x+9+x^2+2x+1=16$ $2x^2-4x-6=0$ $x^2-2x-3=0$ $(x+1)(x-3)=0$ $x=-1, x=3$ Sub x values into (2) $x=-1, y=0$ $x=3, y=4$ Points of intersections are $(-1,0),(3,4)$.	2 Marks Correct solution 1 Mark Correct Cartesian equation of circle and substitutes $y=x+1$ into it
Q12aiii		2 Marks Correct solution 1 Mark Correct graph for the line or the circle with all key features shown

Q12aiv	Area of minor segment is the area of a quarter of a circle take away the area of the triangle. $\frac{1}{4}\pi\times 4^2-\frac{1}{2}\times 4^2=(4\pi-8)\ units^2$	1 Mark Correct solution
Q12bi	$\frac{11x+10}{(x^2+4)(x-2)}=\frac{Ax+B}{x^2+4}+\frac{C}{x-2}$ $(Ax+B)(x-2)+C(x^2+4)=11x+10$ Let $x=2$, $8C=22+10$ $C=4$ Let $x=0$ $-2B+4C=10$ $-2B=-6$ $B=3$ Let $x=1$ $(A+B)\times(-1)+5C=11+10$ $-(A+3)+20=21$ $A=-4$ $\frac{11x+10}{(x^2+4)(x-2)}=\frac{-4x+3}{x^2+4}+\frac{4}{x-2}$	2 Marks Correct solution 1 Mark Finds correct value of A or B or C
Q12bii	$\int_3^4\frac{11x+10}{(x^2+4)(x-2)}dx$ $=\int_3^4\left(\frac{-4x+3}{x^2+4}+\frac{4}{x-2}\right)dx$ $=-2\int_3^4\frac{2x}{x^2+4}dx+3\int_3^4\frac{1}{x^2+4}dx+4\int_3^4\frac{1}{x-2}dx$ $=\left[-2\ln\lvert x^2+4\rvert+\frac{3}{2}\tan^{-1}\frac{x}{2}+4\ln\lvert x-2\rvert\right]_3^4$ $=\left(-2\ln\lvert 4^2+4\rvert+\frac{3}{2}\tan^{-1}\frac{4}{2}+4\ln\lvert 4-2\rvert\right)$ $-\left(-2\ln\lvert 3^2+4\rvert+\frac{3}{2}\tan^{-1}\frac{3}{2}+4\ln\lvert 3-2\rvert\right)$ $=\left(-2\ln 20+\frac{3}{2}\tan^{-1}2+4\ln 2\right)-\left(-2\ln 13+\frac{3}{2}\tan^{-1}\frac{3}{2}\right)$ $=2\ln\left(\frac{13\times 4}{20}\right)+\frac{3}{2}\left(\tan^{-1}2-\tan^{-1}\frac{3}{2}\right)$ $=2\ln\left(\frac{13}{5}\right)+\frac{3}{2}\left(\tan^{-1}2-\tan^{-1}\frac{3}{2}\right)$	2 Marks Correct solution 1 Mark Correct primitive function
Q12ci	$\int\frac{2}{\sqrt{8x-12-x^2}}dx$ $=\int\frac{2}{\sqrt{4-(x^2-8x+16)}}dx$ $=\int\frac{2}{\sqrt{4-(x-4)^2}}dx$ $=\int\frac{2}{2\sqrt{1-\frac{(x-4)^2}{4}}}dx$ $=2\sin^{-1}\left(\frac{x-4}{2}\right)+C$	2 Marks Correct solution 1 Mark Complete the square to obtain $\frac{2}{\sqrt{4-(x-4)^2}}$

Q12cii	$I=\int \sqrt{2x}\ln x\,dx$ $u=\ln x \qquad v'=\sqrt{2}x^{\frac{1}{2}}$ $u'=\frac{1}{x} \qquad v=\frac{2\sqrt{2}x^{\frac{3}{2}}}{3}$ $I=\frac{2\sqrt{2}x^{\frac{3}{2}}\ln x}{3}-\int\frac{1}{x}\times\frac{2\sqrt{2}x^{\frac{3}{2}}}{3}dx$ $I=\frac{2\sqrt{2}x^{\frac{3}{2}}\ln x}{3}-\frac{2\sqrt{2}}{3}\int x^{\frac{1}{2}}dx$ $I=\frac{2\sqrt{2}x^{\frac{3}{2}}\ln x}{3}-\frac{2\sqrt{2}}{3}\times\frac{2x^{\frac{3}{2}}}{3}+C$ $I=\frac{2\sqrt{2}x^{\frac{3}{2}}}{3}\left(\ln x-\frac{2}{3}\right)+C$	2 Marks Correct solution 1 Mark Correctly applies integration by parts
Q13ai	Since $P(x)$ is real, if $a+bi,-3a+bi$ is a root, then $a-bi,-3a-bi$ are also roots of $P(x)$. Sum of roots $(a+bi)+(a-bi)+(-3a+bi)+(-3a+bi)=8$ $-4a=8$ $a=-2$ Product of roots $(a+bi)(a-bi)(-3a+bi)(-3a+bi)=1769$ $(a^2+b^2)(9a^2+b^2)=1769$ $(4+b^2)(36+b^2)=1769$ $144+4b^2+36b^2+b^4=1769$ $b^4+40b^2-1625=0$ $(b^2+65)(b^2-25)=0$ $b^2=25, b^2\neq 65$ since b is real. $b=\pm5$	2 Marks Correct solution 1 Mark Finds a
Q13aii	$P(x)=(x+2+5i)(x+2-5i)(x-6+5i)(x-6-5i)$ For $(x+2+5i)(x+2-5i)$ Sum of roots $(-2+5i)+(-2-5i)=-4$ Product of roots $(-2+5i)(-2-5i)=4+25=29$ $(x+2+5i)(x+2-5i)=x^2+4x+29$ For $(x-6+5i)(x-6-5i)$ Sum of roots $(6+5i)+(6-5i)=12$ Product of roots $(6+5i)(6-5i)=36+25=61$ $(x-6+5i)(x-6-5i)=x^2-12x+61$ $\therefore P(x)=(x^2+4x+29)(x^2-12x+61)$	2 Marks Correct solution 1 Mark Obtains one real quadratic factor
Q13bi	$\overrightarrow{AB}=\left(4\underset{\sim}{i}+3\underset{\sim}{j}+7\underset{\sim}{k}\right)-\left(\underset{\sim}{i}+10\underset{\sim}{k}\right)=3\underset{\sim}{i}+3\underset{\sim}{j}-3\underset{\sim}{k}=3\left(\underset{\sim}{i}+\underset{\sim}{j}-\underset{\sim}{k}\right)$ $\overrightarrow{BC}=\left(8\underset{\sim}{i}+7\underset{\sim}{j}+3\underset{\sim}{k}\right)-\left(4\underset{\sim}{i}+3\underset{\sim}{j}+7\underset{\sim}{k}\right)=4\underset{\sim}{i}+4\underset{\sim}{j}-4\underset{\sim}{k}=4\left(\underset{\sim}{i}+\underset{\sim}{j}-\underset{\sim}{k}\right)$ $\overrightarrow{AB}$ is in the same direction as $\overrightarrow{BC}$, B is a common point, so A,B and C are collinear. Ratio of $AB:BC$ is $3:4$.	2 Marks Correct solution 1 Mark Show they are collinear

Q13bii	$l=\begin{bmatrix}1\\0\\10\end{bmatrix}+\lambda\begin{bmatrix}1\\1\\-1\end{bmatrix}$	1 Mark Correct solution
Q13biii	$\left(4\underset{\sim}{i}+3\underset{\sim}{j}+7\underset{\sim}{k}\right)\cdot\left(\underset{\sim}{i}+\underset{\sim}{j}-\underset{\sim}{k}\right)=4+3-7=0$ $\therefore OB$ is perpendicular to line l.	1 Mark Correct solution
Q13biv	 $A_{\Delta ABC}=\frac{1}{2}\lvert AC\rvert\lvert OB\rvert$ $A_{\Delta ABC}=\frac{1}{2}\left\lvert\left(8\underset{\sim}{i}+7\underset{\sim}{j}+3\underset{\sim}{k}\right)-\left(\underset{\sim}{i}+10\underset{\sim}{k}\right)\right\rvert\times\sqrt{4^2+3^2+7^2}$ $A_{\Delta ABC}=\frac{1}{2}\left\lvert\left(7\underset{\sim}{i}+7\underset{\sim}{j}-7\underset{\sim}{k}\right)\right\rvert\times\sqrt{74}$ $A_{\Delta ABC}=\frac{1}{2}\sqrt{147}\times\sqrt{74}$ $A_{\Delta ABC}=\frac{1}{2}\sqrt{10878}=\frac{7}{2}\sqrt{222}\ units^2$	2 Marks Correct solution 1 Mark Finds $\lvert AC\rvert$ or $\lvert OB\rvert$
Q13ci	$v^2=189-42x-7x^2$ $\frac{1}{2}v^2=\frac{189}{2}-21x-\frac{7}{2}x^2$ $\ddot{x}=\frac{d}{dx}\left(\frac{1}{2}v^2\right)=-21-7x$ $\ddot{x}=-7(x+3)=-\left(\sqrt{7}\right)^2\left(x-(-3)\right)$	1 Mark Correct solution
Q13cii	$n=\sqrt{7}$ $T=\frac{2\pi}{\sqrt{7}}$	1 Mark Correct solution
Q13ciii	$189-42x-7x^2=0$ $x^2+6x-27=0$ $(x+9)(x-3)=0$ $x=-9, x=3$ Centre of motion is -3 Amplitude is 6	2 Marks Correct solution 1 Mark Finds both x values
Q13civ	Maximum speed occurs at centre of motion at $x=-3$. $v^2=189-42\times(-3)-7\times(-3)^2$ $v^2=252$ $v=\pm6\sqrt{7}$ $\therefore$ Maximum speed is $6\sqrt{7}\ m/s$ and occurs at the centre of motion at $x=-3$	1 Mark Correct solution
Q14ai	$u=\frac{\pi}{2}-x$ $du=-dx$ $x=\frac{\pi}{2},\ \ u=0$ $x=0,\ \ u=\frac{\pi}{2}$	2 Marks Correct solution 1 Mark Correct substitution in terms of u

	$\int_0^{\frac{\pi}{2}} \frac{e^{\sin x}}{e^{\sin x}+e^{\cos x}}dx$ $= -\int_{\frac{\pi}{2}}^{0} \frac{e^{\sin\left(\frac{\pi}{2}-u\right)}}{e^{\sin\left(\frac{\pi}{2}-u\right)}+e^{\cos\left(\frac{\pi}{2}-u\right)}}du$ $= \int_0^{\frac{\pi}{2}} \frac{e^{\cos u}}{e^{\cos u}+e^{\sin u}}du$ $= \int_0^{\frac{\pi}{2}} \frac{e^{\cos x}}{e^{\cos x}+e^{\sin x}}dx$	
Q14aii	$\int_0^{\frac{\pi}{2}} \frac{e^{\sin x}}{e^{\sin x}+e^{\cos x}}dx = \int_0^{\frac{\pi}{2}} \frac{e^{\cos x}}{e^{\cos x}+e^{\sin x}}dx$ $\int_0^{\frac{\pi}{2}} \frac{e^{\sin x}}{e^{\sin x}+e^{\cos x}}dx = \frac{1}{2}\left(\int_0^{\frac{\pi}{2}} \frac{e^{\sin x}}{e^{\sin x}+e^{\cos x}}dx + \int_0^{\frac{\pi}{2}} \frac{e^{\cos x}}{e^{\cos x}+e^{\sin x}}dx\right)$ $\int_0^{\frac{\pi}{2}} \frac{e^{\sin x}}{e^{\sin x}+e^{\cos x}}dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{e^{\sin x}+e^{\cos x}}{e^{\sin x}+e^{\cos x}}dx$ $\int_0^{\frac{\pi}{2}} \frac{e^{\sin x}}{e^{\sin x}+e^{\cos x}}dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} 1dx$ $\int_0^{\frac{\pi}{2}} \frac{e^{\sin x}}{e^{\sin x}+e^{\cos x}}dx = \frac{1}{2}[x]_0^{\frac{\pi}{2}}$ $\int_0^{\frac{\pi}{2}} \frac{e^{\sin x}}{e^{\sin x}+e^{\cos x}}dx = \frac{1}{2}\left[\frac{\pi}{2}-0\right]$ $\int_0^{\frac{\pi}{2}} \frac{e^{\sin x}}{e^{\sin x}+e^{\cos x}}dx = \frac{\pi}{4}$	1 Mark Correct solution
Q14b	Suppose $\log_n(n-1)$ is rational, then $\log_n(n-1) = \frac{p}{q}$ where p and q are integers with no common factors, and $q \neq 0$ $n-1 = n^{\frac{p}{q}}$ $(n-1)^q = n^p$ If n is even, then RHS is even, LHS would be odd, which is a contradiction. If n is odd, then RHS is odd, LHS would be even, which is also a contradiction. $\therefore \log_n(n-1)$ is irrational	2 Marks Correct solution 1 Mark Makes significant progress
Q14ci	$z^n + \frac{1}{z^n} = e^{in\theta} + e^{-in\theta}$ $z^n + \frac{1}{z^n} = \cos(n\theta) + i\sin(n\theta) + \cos(-n\theta) + i\sin(-n\theta)$ $z^n + \frac{1}{z^n} = \cos(n\theta) + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta)$ $z^n + \frac{1}{z^n} = 2\cos(n\theta)$	1 Mark Correct solution
Q14cii	$5z^4 - 11z^3 + 16z^2 - 11z + 5 = 0$ Divide both sides by z^2 $5z^2 - 11z + 16 - 11z^{-1} + 5z^{-2} = 0$ $5(z^2 + z^{-2}) - 11(z + z^{-1}) + 16 = 0$ $5 \times 2\cos(2\theta) - 11 \times 2\cos\theta + 16 = 0$ $10(2\cos^2\theta - 1) - 22\cos\theta + 16 = 0$ $20\cos^2\theta - 10 - 22\cos\theta + 16 = 0$ $20\cos^2\theta - 22\cos\theta + 6 = 0$	4 Marks Correct solution 3 Marks Finds all values of $\cos\theta$ 2 Marks Generate

	$10\cos^2\theta - 11\cos\theta + 3 = 0$ $(2\cos\theta - 1)(5\cos\theta - 3) = 0$ $\cos\theta = \frac{1}{2}, \ \cos\theta = \frac{3}{5}$ $\sin\theta = \pm\frac{\sqrt{3}}{2}, \ \sin\theta = \pm\frac{4}{5}$ $\therefore z = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i, \ \frac{3}{5} \pm \frac{4}{5}i$	$10\cos^2\theta$ $- 11\cos\theta + 3 = 0$ 1 Mark Generate $5(z^2 + z^{-2})$ $- 11(z + z^{-1})$ $+ 16 = 0$
Q14di	$I_n = \int \frac{dx}{(x^2+1)^n}$ $u = (x^2+1)^{-n} \qquad v' = 1$ $u = -2xn(x^2+1)^{-(n+1)} \qquad v = x$ $I_n = \frac{x}{(x^2+1)^n} - \int \frac{-2x^2ndx}{(x^2+1)^{n+1}}$ $I_n = \frac{x}{(x^2+1)^n} + 2n\int \frac{x^2dx}{(x^2+1)^{n+1}}$ $I_n = \frac{x}{(x^2+1)^n} + 2n\int \frac{(x^2+1-1)dx}{(x^2+1)^{n+1}}$ $I_n = \frac{x}{(x^2+1)^n} + 2n\int \left(\frac{(x^2+1)}{(x^2+1)^{n+1}} - \frac{1}{(x^2+1)^{n+1}}\right)dx$ $I_n = \frac{x}{(x^2+1)^n} + 2n\int \left(\frac{1}{(x^2+1)^{n}} - \frac{1}{(x^2+1)^{n+1}}\right)dx$ $I_n = \frac{x}{(x^2+1)^n} + 2nI_n - 2nI_{n+1}$ $2nI_{n+1} = \frac{x}{(x^2+1)^n} + 2nI_n - I_n$ $2nI_{n+1} = \frac{x}{(x^2+1)^n} + (2n-1)I_n$ $I_{n+1} = \frac{1}{2n}\left[\frac{x}{(x^2+1)^n} + (2n-1)I_n\right]$ Replace n with $n-1$ $I_n = \frac{1}{2(n-1)}\left[\frac{x}{(x^2+1)^{n-1}} + (2(n-1)-1)I_{n-1}\right]$ $I_n = \frac{1}{2(n-1)}\left[\frac{x}{(x^2+1)^{n-1}} + (2n-3)I_{n-1}\right]$	3 Marks Correct solution 2 Marks Makes significant progress 1 Mark Correct integration by parts to obtain $\frac{x}{(x^2+1)^n}$ $- \int \frac{-2x^2ndx}{(x^2+1)^{n+1}}$
Q14dii	$I_1 = \int_0^1 \frac{dx}{(x^2+1)}$ $I_1 = [\tan^{-1}x]_0^1$ $I_1 = \tan^{-1}1 - \tan^{-1}0$ $I_1 = \frac{\pi}{4}$ $I_2 = \int_0^1 \frac{dx}{(x^2+1)^2}$ $I_2 = \frac{1}{2(2-1)}\left[\left[\frac{x}{(x^2+1)^{2-1}}\right]_0^1 + (2\times 2-3)I_{2-1}\right]$ $I_2 = \frac{1}{2}\left[\left(\frac{1}{(1^2+1)}\right) + I_1\right]_0^1$ $I_2 = \frac{1}{2}\left[\frac{1}{2} + \frac{\pi}{4}\right]$ $I_2 = \frac{1}{4} + \frac{\pi}{8}$	2 Marks Correct solution 1 Mark Finds I_1

Q15a	$t=\tan\frac{x}{2}$ $dt=\frac{1}{2}\sec^2\frac{x}{2}dx$ $2dt=\left(1+\tan^2\frac{x}{2}\right)dx$ $dx=\frac{2}{1+t^2}dt$ $x=0, t=0$ $x=\frac{\pi}{2}, t=1$ $\cos x=\frac{1-t^2}{1+t^2}$ $I=\int_0^{\frac{\pi}{2}}\frac{2}{4+5\cos x}dx$ $I=\int_0^1\frac{2}{4+5\times\frac{1-t^2}{1+t^2}}\times\frac{2}{1+t^2}dt$ $I=\int_0^1\frac{4}{4(1+t^2)+5(1-t^2)}dt$ $I=\int_0^1\frac{4}{9-t^2}dt$ $I=\int_0^1\frac{4}{(3+t)(3-t)}dt$ $\frac{4}{(3+t)(3-t)}=\frac{A}{3+t}+\frac{B}{3-t}$ $A(3-t)+B(3+t)=4$ Let $t=3$ $6B=4$ $B=\frac{2}{3}$ Let $t=-3$ $6A=4$ $A=\frac{2}{3}$ $I=\frac{2}{3}\int_0^1\left(\frac{1}{3+t}+\frac{1}{3-t}\right)dt$ $I=\frac{2}{3}[\ln\lvert 3+t\rvert-\ln\lvert 3-t\rvert]_0^1$ $I=\frac{2}{3}[(\ln\lvert 3+1\rvert-\ln\lvert 3-1\rvert)-(\ln\lvert 3+0\rvert-\ln\lvert 3-0\rvert)]$ $I=\frac{2}{3}\ln 2$	4 Marks Correct solution 3 Marks Correct primitive function in terms of t 2 Marks Correct partial fraction 1 Mark Correct substitution
Q15bi	$v=4e^{-2x}-3$ $v^2=16e^{-4x}-24e^{-2x}+9$ $\frac{1}{2}v^2=8e^{-4x}-12e^{-2x}+\frac{9}{2}$ $\ddot{x}=\frac{d}{dx}\left(\frac{1}{2}v^2\right)=-32e^{-4x}+24e^{-2x}$	2 Marks Correct solution 1 Mark Correct $\ddot{x}$
	Initially $x=0$ $\ddot{x}=-32e^0+24e^0$ $\ddot{x}=-8$	
Q15bii	$v=\frac{dx}{dt}=4e^{-2x}-3$ $\frac{dx}{dt}=\frac{4}{e^{2x}}-3=\frac{4-3e^{2x}}{e^{2x}}$ $\frac{dt}{dx}=\frac{e^{2x}}{4-3e^{2x}}$ $t=\int\frac{e^{2x}}{4-3e^{2x}}dx$ $t=-\frac{1}{6}\ln\lvert 4-3e^{2x}\rvert+C$ $t-C=-\frac{1}{6}\ln\lvert 4-3e^{2x}\rvert$ $C-t=\frac{1}{6}\ln\lvert 4-3e^{2x}\rvert$ $4-3e^{2x}=Ae^{-6t}$ $t=0, x=0$ $4-3e^0=A\times e^0$ $A=1$ $4-3e^{2x}=e^{-6t}$ $3e^{2x}=4-e^{-6t}$ $e^{2x}=\frac{1}{3}\left(\frac{4e^{6t}-1}{e^{6t}}\right)$ $2x=\ln\left[\frac{1}{3}\left(\frac{4e^{6t}-1}{e^{6t}}\right)\right]$ $x=\frac{1}{2}\ln\left[\frac{1}{3}\left(\frac{4e^{6t}-1}{e^{6t}}\right)\right]$	3 Marks Correct solution 2 Marks Finds $t=-\frac{1}{6}\ln\lvert 4-3e^{2x}\rvert+C$ 1 Mark Finds $\frac{dt}{dx}=\frac{e^{2x}}{4-3e^{2x}}$
Q15biii	$e^{2x}=\frac{1}{3}\left(\frac{4e^{6t}-1}{e^{6t}}\right)$ $e^{-2x}=3\left(\frac{e^{6t}}{4e^{6t}-1}\right)$ $v=4e^{-2x}-3$ $v=4\times 3\left(\frac{e^{6t}}{4e^{6t}-1}\right)-3$ $v=3\left(\frac{4e^{6t}}{4e^{6t}-1}-1\right)$ $v=3\left(\frac{4e^{6t}}{4e^{6t}-1}-\frac{4e^{6t}-1}{4e^{6t}-1}\right)$ $v=\frac{3}{4e^{6t}-1}$	1 Mark Correct solution
Q15ci	$\left(\sqrt{a}-\sqrt{b}\right)^2\geq 0$ $a-2\sqrt{ab}+b\geq 0$ $a+b\geq 2\sqrt{ab}$ $\therefore\frac{a+b}{2}\geq\sqrt{ab}$	1 Mark Correct solution

Q15cii	$\frac{\frac{a+b}{2}+\frac{c+d}{2}}{2} \ge \sqrt{\left(\frac{a+b}{2}\right)\left(\frac{c+d}{2}\right)}$ $\frac{a+b+c+d}{4} \ge \sqrt{\frac{ac+ad+bc+bd}{4}}$ $\frac{a+b+c+d}{4} \ge \frac{\sqrt{ac+ad+bc+bd}}{2}$ $a+b+c+d \ge 2\sqrt{ac+ad+bc+bd}$ $\therefore (a+b+c+d)^2 \ge 4(ac+bc+bd+ad) \dots\dots (1)$ Alternatively Let $a=a+b, b=c+d$ $\frac{a+b}{2} \ge \sqrt{ab}$ $\frac{a+b+c+d}{2} \ge \sqrt{(a+b)(c+d)}$ $\frac{(a+b+c+d)^2}{4} \ge (a+b)(c+d)$ $\frac{(a+b+c+d)^2}{4} \ge (ac+bc+bd+ad)$ $\therefore (a+b+c+d)^2 \ge 4(ac+bc+bd+ad)$	2 Marks Correct solution 1 Mark Shows $\frac{\frac{a+b}{2}+\frac{c+d}{2}}{2} \ge \sqrt{\left(\frac{a+b}{2}\right)\left(\frac{c+d}{2}\right)}$
Q15ciii	Similarly $\frac{\frac{a+c}{2}+\frac{b+d}{2}}{2} \ge \sqrt{\left(\frac{a+c}{2}\right)\left(\frac{b+d}{2}\right)}$ $(a+b+c+d)^2 \ge 4(ab+ad+bc+cd) \dots\dots (2)$ $\frac{\frac{a+d}{2}+\frac{b+c}{2}}{2} \ge \sqrt{\left(\frac{a+d}{2}\right)\left(\frac{b+c}{2}\right)}$ $(a+b+c+d)^2 \ge 4(ab+ac+bd+cd) \dots\dots (3)$ $(1)+(2)+(3)$ $3(a+b+c+d)^2$ $= 4[(ac+bc+bd+ad)+(ab+ad+bc+cd)$ $+(ab+ac+bd+cd)]$ $3(a+b+c+d)^2 = 4(2ab+2ad+2bc+2cd+2bd+2ac)$ $(a+b+c+d)^2 = \frac{8}{3}(ab+ad+bc+cd+bd+ac)$	2 Marks Correct solution 1 Mark Makes significant progress
Q16ai	$0<x<1$ $1<x+1<2$ $\frac{1}{1} > \frac{1}{x+1} > \frac{1}{2}$ $\frac{1}{2} < \frac{1}{x+1} < 1$ Multiply both sides by $x(1-x)^3$ $\frac{1}{2}x(1-x)^3 < \frac{x(1-x)^3}{1+x} < x(1-x)^3$ Since $x>0$ and $(1-x)^3>0$ for $0<x<1$, so the inequalities hold.	2 Marks Correct solution 1 Mark Makes significant progress

Q16aii	$\frac{1}{2}x(1-x)^3 < \frac{x(1-x)^3}{1+x} < x(1-x)^3$ Since all three functions are positive for $0<x<1$, then areas under the curve also follows the same inequalities. $\frac{1}{2}\int_0^1 x(1-x)^3dx < \int_0^1 \frac{x(1-x)^3}{1+x}dx < \int_0^1 x(1-x)^3dx$	1 Mark Correct solution
Q16aiii	$I = \int_0^1 x(1-x)^3dx$ $u=1-x$ $du=-dx$ $x=1, u=0$ $x=0, u=1$ $I = \int_1^0 -(1-u)u^3dx$ $I = \int_0^1 (u^3-u^4)dx$ $I = \left[\frac{u^4}{4}-\frac{u^5}{5}\right]_0^1$ $I = \frac{1}{4}-\frac{1}{5}$ $I = \frac{1}{20}$ $\frac{1}{2}\int_0^1 x(1-x)^3dx = \frac{1}{40}$ $\frac{1}{2}\int_0^1 x(1-x)^3dx < \int_0^1 \frac{x(1-x)^3}{1+x}dx < \int_0^1 x(1-x)^3dx$ $\frac{1}{40} < \frac{67}{12} - 8\ln 2 < \frac{1}{20}$ $-\frac{667}{120} < -8\ln 2 < -\frac{83}{15}$ Divide both sides by -8, inequality sign changes $\frac{667}{960} > \ln 2 > \frac{83}{120}$ $\therefore \frac{83}{120} < \ln 2 < \frac{667}{960}$	3 Marks Correct solution 2 Marks Makes significant progress 1 Mark Finds $\int_0^1 x(1-x)^3dx$ or $\frac{1}{2}\int_0^1 x(1-x)^3dx$
Q16bi	$b_2 = b_1(b_1+1) = 1\times 2 = 2$ $b_3 = b_2(b_2+1) = 2\times 3 = 6$ $b_4 = b_3(b_3+1) = 6\times 7 = 42$	1 Mark Correct solution
Q16bii	$b_{n+1} = 1 + \sum_{r=1}^{n}(b_r)^2 = 1 + (b_1)^2 + (b_2)^2 + (b_3)^2 + \cdots + (b_n)^2$ 1. Prove statement is true for $n=1$ $LHS = b_2 = 2$ $RHS = 1 + (b_1)^2 = 1 + 1^2 = 2$ $LHS = RHS$ $\therefore$Statement is true for $n=1$.	3 Marks Correct solution 2 Marks Makes significant progress 1 Mark

	2. Assume statement is true for $n = k$, k some positive integer $b_{k+1} = 1 + \sum_{r=1}^{k} (b_r)^2 = 1 + (b_1)^2 + (b_2)^2 + (b_3)^2 + \cdots + (b_k)^2$ 3. Prove statement is true for $n = k + 1$. $b_{k+2} = 1 + \sum_{r=1}^{k+1} (b_r)^2 = 1 + (b_1)^2 + (b_2)^2 + (b_3)^2 + \cdots + (b_k)^2 + (b_{k+1})^2$ $LHS = b_{k+2}$ $LHS = b_{k+1}(b_{k+1} + 1)$ $LHS = (b_{k+1})^2 + b_{k+1}$ $LHS = (b_{k+1})^2 + 1 + \sum_{r=1}^{k} (b_r)^2$ $LHS = (b_{k+1})^2 + 1 + (b_1)^2 + (b_2)^2 + (b_3)^2 + \cdots + (b_k)^2$ $LHS = 1 + (b_1)^2 + (b_2)^2 + (b_3)^2 + \cdots + (b_k)^2 + (b_{k+1})^2$ $LHS = 1 + \sum_{r=1}^{k+1} (b_r)^2$ $LHS = RHS$ $\therefore$ statement is true by mathematical induction for all integer $n \geq 1$	Proves for the initial case
Q16biii	$b_5 = b_4(b_4 + 1) = 42 \times 43 = 1806$ $b_5 = 1 + (b_1)^2 + (b_2)^2 + (b_3)^2 + (b_4)^2$ $b_5 = 1^2 + 1^2 + 2^2 + 6^2 + 42^2 = 1806$	1 Mark Correct solution
Q16biv	$RTP: (2b_{n+1} + 1)^2 = (2b_n + 1)^2 + (2b_{n+1})^2$ $LHS = (2b_{n+1} + 1)^2$ $LHS = (2b_{n+1})^2 + 4b_{n+1} + 1$ $LHS = (2b_{n+1})^2 + 4b_n(b_n + 1) + 1$ $LHS = (2b_{n+1})^2 + 4(b_n)^2 + 4b_n + 1$ $LHS = (2b_{n+1})^2 + (2b_n + 1)^2$ $LHS = RHS$ $\therefore (2b_{n+1} + 1)^2 = (2b_n + 1)^2 + (2b_{n+1})^2$	1 Mark Correct solution
Q16bv	$RTP: (2b_{n+1} + 1)^2 = (2b_1 + 1)^2 + \sum_{r=2}^{n+1} (2b_r)^2$ $LHS = (2b_{n+1} + 1)^2$ $LHS = (2b_n + 1)^2 + (2b_{n+1})^2$ $LHS = (2b_{n-1} + 1)^2 + (2b_n)^2 + (2b_{n+1})^2$ $LHS = (2b_{n-2} + 1)^2 + (2b_{n-1})^2 + (2b_n)^2 + (2b_{n+1})^2$ … $LHS = (2b_1 + 1)^2 + (2b_2)^2 + (2b_3)^2 + (2b_4)^2 + \cdots + (2b_n)^2 + (2b_{n+1})^2$ $LHS = (2b_1 + 1)^2 + \sum_{r=2}^{n+1} (2b_r)^2$ $\therefore (2b_{n+1} + 1)^2 = (2b_1 + 1)^2 + \sum_{r=2}^{n+1} (2b_r)^2$	2 Marks Correct solution 1 Mark Makes significant progress
Q16bvi	$(2b_5 + 1)^2 = (2b_1 + 1)^2 + \sum_{r=2}^{5} (2b_r)^2$ $(2 \times 1806 + 1)^2 = (2 \times 1 + 1)^2 + (2b_2)^2 + (2b_3)^2 + (2b_4)^2 + (2b_5)^2$ $3613^2 = 3^2 + (2 \times 2)^2 + (2 \times 6)^2 + (2 \times 42)^2 + (2 \times 1806)^2$ $3613^2 = 3^2 + 4^2 + 12^2 + 84^2 + 3612^2$	1 Mark Correct solution