Chebyshev Polynomials in Past Papers Solved via Hypergeometric Functions

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Sometimes we see questions like find $\cos(n\theta)$ as a function of $\cos\theta$. These are Chebyshev polynomials of the first kind. One may do it using de Moivre's theorem, but that would be quite lengthy. A quicker way is to use a hypergeometric function which gives the answer in 1 step. If the question however was to find $\frac{\sin(n\theta)}{\sin\theta}$ as a function of $\cos\theta$, that is a Chebyshev polynomial of the second kind, which can also be done with hypergeometric functions, but is a little trickier.

Here are some questions from past Leaving Certificate exams and HSC exams which can be done much more quickly with hypergeometric functions, rather than using de Moivre's theorem.

1916. Find $\cos A$ in terms of $\cos\frac{A}{3}$

1920. Find $\frac{\sin 9\theta}{\sin\theta}$ in terms of x where $x = 2\cos\theta$

1925. Find $\cos 7\theta$ in terms of $\cos\theta$

1926. Find $\cos 5\theta$ in terms of $\cos\theta$

1945. Find $\cos 6\theta$ in terms of $\cos\theta$

1948. Express $2\cos 10\theta$ as a sum of powers of $2\cos\theta$

2016. Find $\cos 4\theta$ in terms of $\cos\theta$

Answers are on the next page.

Hint: $\cos n\theta = {}_2F_1(-n, n; \frac{1}{2}; \frac{1-\cos\theta}{2})$ and $\frac{\sin n\theta}{\sin\theta} = n \cdot {}_2F_1(1-n, 1+n; \frac{3}{2}; \frac{1-\cos\theta}{2})$

Solutions

1916 Leaving Certificate Mathematics I Honours Q7

$$\cos A = \cos 3(\tfrac{A}{3}) = {}_2F_1\left(-3, 3; \tfrac{1}{2}; \tfrac{1-\cos\frac{A}{3}}{2}\right) = 4\cos^3\tfrac{A}{3} - 3\cos\tfrac{A}{3}$$

1920 Leaving Certificate Mathematics I Honours Q8

$$\tfrac{\sin 9\theta}{\sin\theta} = 9{}_2F_1(-8, 10; \tfrac{3}{2}; \tfrac{1}{2}(1 - \tfrac{x}{2})) = (x^2-1)(x^6 - 6x^4 + 9x^2 - 1) \text{ where } x = 2\cos\theta$$

1925 Leaving Certificate Mathematics I Honours Q11

$$\cos 7\theta = {}_2F_1\left(-7, 7; \tfrac{1}{2}; \tfrac{1-\cos\theta}{2}\right) = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$$

1926 Leaving Certificate Mathematics I Honours Q9

$$\cos 5\theta = {}_2F_1\left(-5, 5; \tfrac{1}{2}; \tfrac{1-\cos\theta}{2}\right) = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

1945 Leaving Certificate Mathematics I Honours Q10 paper 2

$$\cos 6\theta = {}_2F_1\left(-6, 6; \tfrac{1}{2}; \tfrac{1-\cos\theta}{2}\right) = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$$

1948 Leaving Certificate Mathematics I Honours Q12 paper 2

$$\begin{aligned} 2\cos 10\theta &= 2{}_2F_1\left(-10, 10; \tfrac{1}{2}; \tfrac{1-\cos\theta}{2}\right) \\ &= 1024\cos^{10}\theta - 2560\cos^8\theta + 2240\cos^6\theta - 800\cos^4\theta + 100\cos^2\theta - 2 \\ &= (2\cos\theta)^{10} - 10(2\cos\theta)^8 + 35(2\cos\theta)^6 - 50(2\cos\theta)^4 + 25(2\cos\theta)^2 - 2 \end{aligned}$$

2016 HSC Mathematics Extension 2 Q12cii

$$\cos 4\theta = {}_2F_1\left(-4, 4; \tfrac{1}{2}; \tfrac{1-\cos\theta}{2}\right) = 8\cos^4\theta - 8\cos^2\theta + 1$$