SYDNEY BOYS HIGH SCHOOL

NESA Number:

Name: ____________________

Class: ____________________

2020 | YEAR 12 TERM 3 TRIAL HSC

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided with this paper
- Marks may **NOT** be awarded for messy or badly arranged work
- For questions in Section II, show ALL relevant mathematical reasoning and/or calculations or

Total Marks: 70

Section I – 10 marks (pages 2 – 6)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 8 – 12)

- Attempt all Questions in Section II
- Allow about 1 hour and 45 minutes for this section

Examiner: AMG

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10

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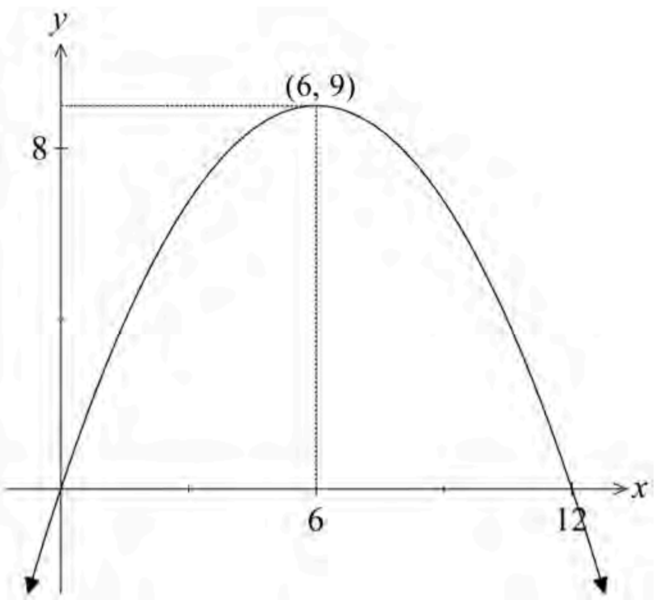
1 The polynomial $P(x) = x^3 + 2x + k$ has $(x-2)$ as a factor.
What is the value of k?

A. -12 B. -10

C. 10 D. 12

2



Which of the parametric equations below represents the parabola above?

A. $x = 12t,\ y = 9t^2$ B. $x = 12t,\ y = 9 - t^2$

C. $x = 6 - 2t,\ y = 9 - t^2$ D. $x = 2t - 6,\ y = 9t - t^2$

3 What is the domain and range of $y = 2\cos^{-1}(x-1)$?

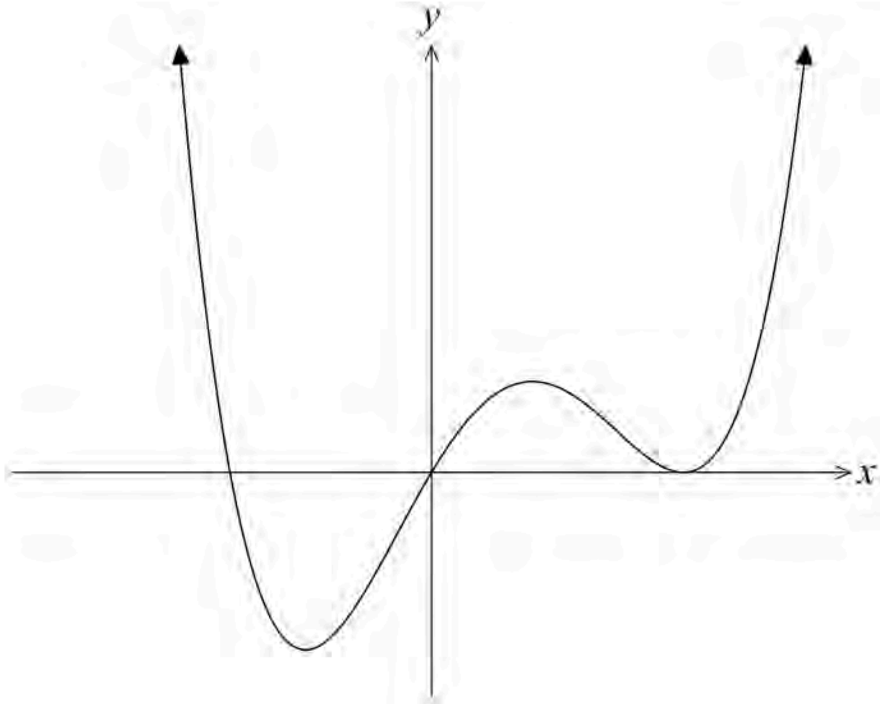
A. Domain: $0 \le x \le 2$. Range: $0 \le y \le \pi$

B. Domain: $-1 \le x \le 1$. Range: $0 \le y \le \pi$

C. Domain: $-1 \le x \le 1$. Range: $0 \le y \le 2\pi$

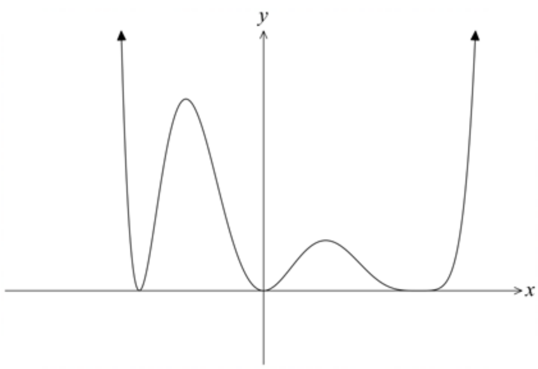
D. Domain: $0 \le x \le 2$. Range: $0 \le y \le 2\pi$

4 The diagram shows the graph of $y = f(x)$.

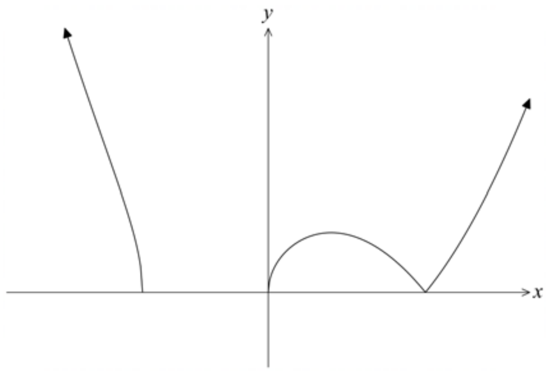


Which of the following best shows the graph of $y^2 = f(x)$?

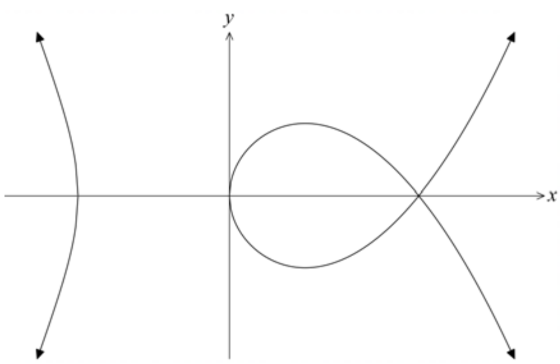
A.



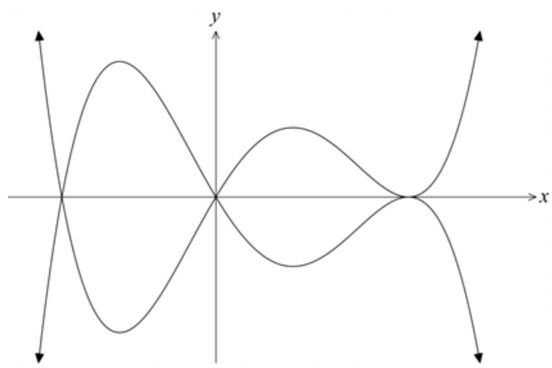
B.



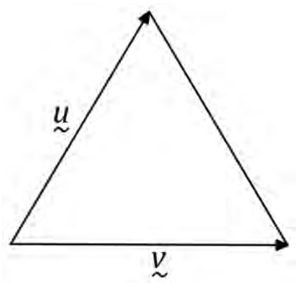
C.



D.



5 An equilateral triangle of side 3 units is shown below.
Vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are represented in the diagram below.



What is the value of $\underset{\sim}{u} \cdot \underset{\sim}{v}$?

A. 0

B. $\dfrac{9}{\sqrt{2}}$

C. $\dfrac{9}{2}$

D. 9

6 How many different ways can 4 people be chosen from a group of 20 people?

A. 4 845

B. 160 000

C. 116 280

D. 240 000

7 The direction (slope) field for a certain first order differential equation is shown below.

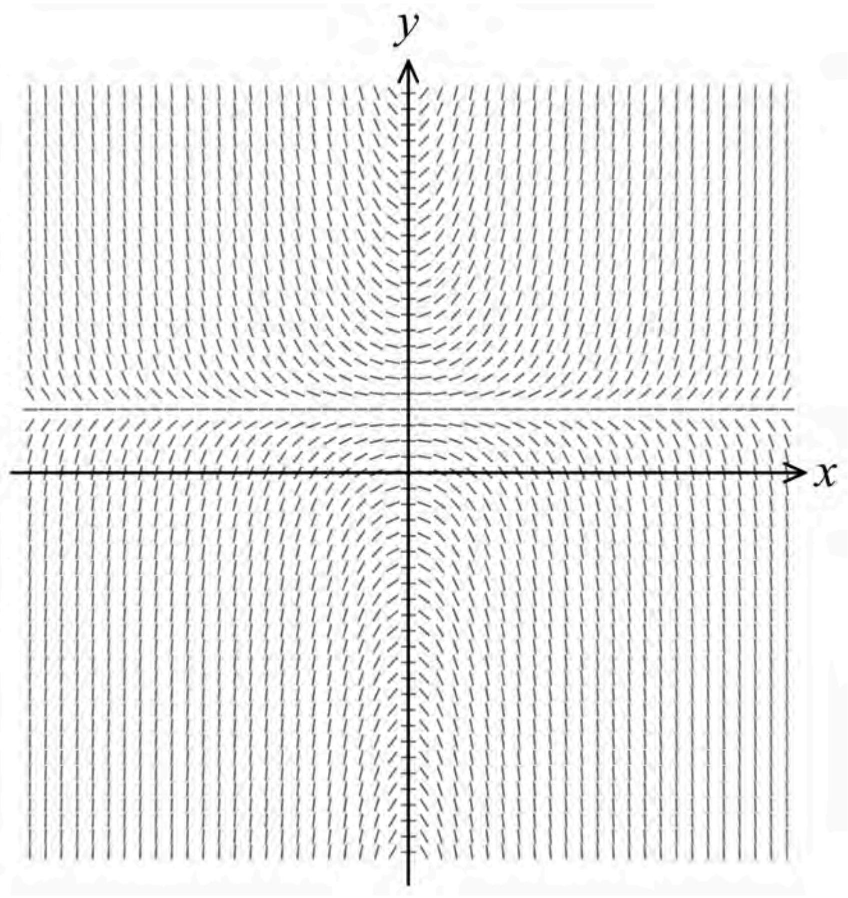
Which of the following below is the differential equation?

A. $\dfrac{dy}{dx} = x^2 + y^2$

B. $\dfrac{dy}{dx} = x^2 - y^2$

C. $\dfrac{dy}{dx} = xy - x$

D. $\dfrac{dy}{dx} = y^2 - x^2$



8 The region bounded by the curve $y = e^{\frac{x}{2}}$ and the y-axis for $1 \le y \le 2$ is rotated about the y-axis to form a solid.

Which of the following integrals would give the volume of the solid?

A. $V = 4\pi \int_1^2 y \ln y \, dy$

B. $V = 4\pi \int_1^2 \ln y^2 \, dy$

C. $V = 4\pi \int_1^2 xe^x \, dx$

D. $V = 4\pi \int_1^2 (\ln y)^2 \, dy$

9 Which of the following is the vector projection of $\underset{\sim}{p}$ onto $\underset{\sim}{q}$,

where $\underset{\sim}{p} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and $\underset{\sim}{q} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$?

A. $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$

B. $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$

C. $\begin{pmatrix} -2\sqrt{5} \\ 4\sqrt{5} \end{pmatrix}$

D. $\begin{pmatrix} 2\sqrt{5} \\ -4\sqrt{5} \end{pmatrix}$

10 Which of the following is an expression for $\int \frac{x}{\sqrt{9 - x^2}} \, dx$?

A. $-\sqrt{9 - x^2} + C$

B. $-2\sqrt{9 - x^2} + C$

C. $\sqrt{9 - x^2} + C$

D. $2\sqrt{9 - x^2} + C$

Section II

60 marks
Attempt Questions 11–14
Allow 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE Writing Booklet.

(a) Six chairs are to be painted. Two of the chairs are to be painted black. **1**
The other four are to be painted blue, green, yellow and grey.
The chairs are then arranged in a row.
How many possible arrangements are there for the colours of the chairs?

(b) (i) Find $\dfrac{dy}{dx}$ if $y = xe^{-x}$. **2**

(ii) Hence find $\displaystyle\int_0^1 xe^{-x}\,dx$. **3**

(c) Solve $\dfrac{3}{x(2x-1)} \geq 1$? **3**

(d) The position vectors for the points P, Q, R and S are $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ and $\mathbf{s}$ respectively.
Shown below in column vector notation.

$$\mathbf{p} = \begin{pmatrix} -8 \\ -8 \end{pmatrix},\ \mathbf{q} = \begin{pmatrix} 3 \\ 6 \end{pmatrix},\ \mathbf{r} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \text{ and } \mathbf{s} = \begin{pmatrix} -5 \\ 7 \end{pmatrix}$$

Find

(i) $\overrightarrow{PQ}$ **1**

(ii) $-\overrightarrow{PQ} - \overrightarrow{RS}$ **1**

(e) Find the coefficient of x in the expansion of $\left(2x - \dfrac{1}{2x}\right)^9$. **2**

(f) Differentiate $y = e^x \arcsin(2x)$. **2**

Question 12 (14 marks) Use a SEPARATE Writing Booklet.

(a) Marty the Martian has an infinite number of red, blue, yellow, and black socks in a drawer. If Marty is pulling out socks in the dark, what is the smallest number of socks that Marty must pull out of the drawer to guarantee getting ten socks of the same colour? **1**

(b) Prove by mathematical induction that $4^n + 14$ is divisible by 6 for all positive integers n, $n \geq 1$. **3**

(c) Solve $(n+2)! = 72n!$ **2**

(d) In a class of 27 students, there are 14 boys and 13 girls.

The class needs to elect two boys and two girls for the student council.

(i) In how many ways could the representatives be chosen? **2**

(ii) If the class elected Henry and Olivia for the student council, in how many ways could the representatives now be chosen? **1**

(e) Given the relation $4x^2 + 9y^2 = 36$:

(i) Show that $\dfrac{dy}{dx} = -\dfrac{4x}{9y}$. **1**

(ii) Find the gradient of the tangent at the point on the curve where $x = 2$, and $y > 0$. **1**

(f) Suppose $Q(x) = Ax^3 + Bx^2 + Cx + D$ with $A, B, C, D \in \mathbb{R}$, $A \neq 0$.

(i) What can be deduced about the zeros of $Q(x)$ if $B^2 - 3AC < 0$? **2**

(ii) If $B^2 - 3AC = 0$ and $Q\left(-\dfrac{B}{3A}\right) = 0$, what is the multiplicity of the root $x = -\dfrac{B}{3A}$? **1**

Question 13 (15 marks) Use a SEPARATE Writing Booklet.

(a) A model for the population, K, of kangaroos is

$$K = \frac{21\,000}{7 + 3e^{-\frac{t}{3}}},$$

where t is the time in years from today.

(i) Show that K satisfies the differential equation **2**

$$\frac{dK}{dt} = \frac{1}{3}\left(1 - \frac{K}{3000}\right)K.$$

(ii) What is the population today? **1**

(iii) What does the model predict that the eventual population will be? **1**

(iv) What is the annual percentage rate of growth today? **1**

(b) Solve $\dfrac{\sin x + \cos x}{\sqrt{2}} = \dfrac{\sqrt{3}}{2}$ for $0° \le x \le 360°$. **3**

(c) By use of a suitable sketch, or otherwise, solve $\arcsin(3x + 1) = \arccos x$. **3**

(d) Find the exact value of $\displaystyle\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1 - x^2}}\, dx$, using the substitution $x = \sin\theta$. **4**

Question 14 (16 marks) Use a SEPARATE Writing Booklet.

(a) A golfer hits a golf ball from a point O with speed V m/s at an angle θ above the horizontal, where $0^\circ < \theta < 90^\circ$. The ball just passes over a 2.25 m high tree after 1.5 seconds. The tree is 60 metres away on level ground from the point from which the ball was hit. The acceleration due to gravity is 10 m/s^2.

The position vector, $\underset{\sim}{r}$, at any time, t seconds, is given by

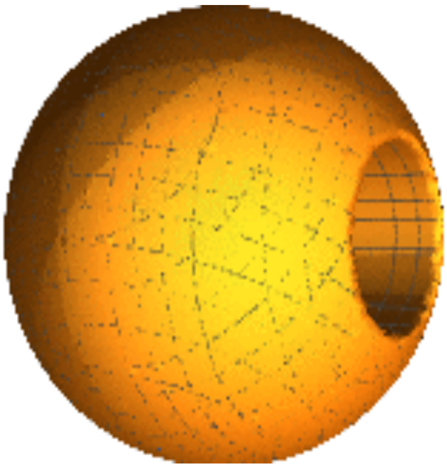
$$\underset{\sim}{r}(t) = Vt\cos\theta\,\underset{\sim}{i} + \left(Vt\sin\theta - 5t^2\right)\underset{\sim}{j}$$ (Do **NOT** prove)

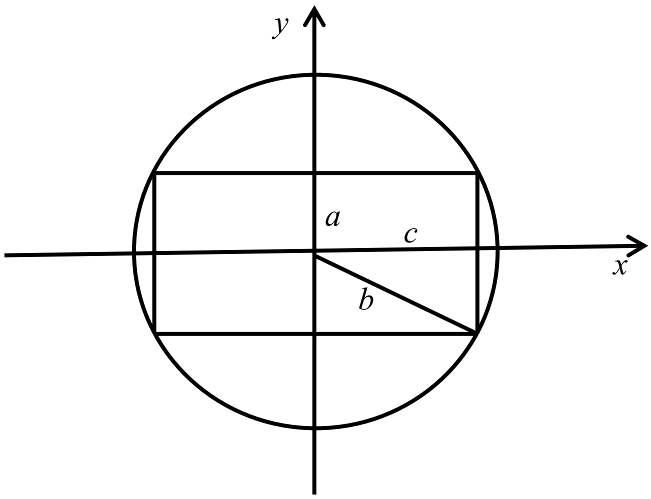
(i) What is the initial angle, θ, of projection of the golf ball, to the nearest minute? **3**

(ii) What is the initial speed (V m/s) of the golf ball. **1**

(iii) How far from the base of the tree does the ball strike the ground? Leave your answer correct to the nearest metre. **1**

(b) To make a macramé bead, a cylindrical hole of radius 5 mm is bored through a sphere of radius 13 mm.
The axis of the cylinder coincides with a diameter of the sphere.





(i) State the values of a, b, and c in the diagram. **2**

(ii) Find the volume of the bead, by using a rotation about the x-axis. **4**

Question 14 continues on page 12

Question 14 (continued)

(c) (i) By considering $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_nx^n$, where $n \in \mathbb{Z}^+$, show that **2**

$$n2^{n-1} = {}^nC_1 + 2\,{}^nC_2 + 3\,{}^nC_3 + ... + n\,{}^nC_n.$$

(ii) By considering the coefficient of x^n in the expansion of **3**

$$\left(1+x+x^2+...+x^n\right)^2(1+x)^n,$$

find the number of combinations of n letters out of $3n$ letters of which n letters are 'a', n letters are 'b' and the rest of the letters are unlike. Leave your answer in simplest exact form.

End of paper

SYDNEY BOYS HIGH SCHOOL

2020

YEAR 12
TERM 3
TRIAL HSC - ASSESSMENT TASK 3

Mathematics Extension 1

Sample Solutions

Quick MC Answers

1	A
2	C
3	D
4	C
5	C
6	A
7	C
8	D
9	B
10	A

NOTE: Before putting in an appeal re marking, first consider that the mark is not linked to the amount of ink you have used.

Just because you have shown 'working' does not justify that your solution is worth any marks.

2020 Y12 X1 THSC Multiple choice solutions

Mean (out of 10): 8.67

1. $P(2) = 8+4+k = 0$

$\therefore k = -12$

(A)

A	149
B	5
C	1
D	4

2. A If $x=12$, $t=1$ $\therefore y=9$ ✗

B If $x=6$, $t=\frac{1}{2}$ $\therefore y = 8\frac{3}{4}$ ✗

C If $x=0$, $t=3$ $\therefore y=0$ OK

If $x=6$, $t=0$ $\therefore y=9$ OK

If $x=12$, $t=-3$ $\therefore y=0$ OK ✓

D If $x=0$, $t=3$ $\therefore y=18$ ✗

(C)

A	3
B	8
C	144
D	4

3. $-1 \le x-1 \le 1$

$\therefore 0 \le x \le 2$

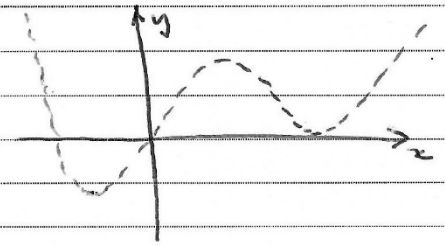
$0 \le \cos^{-1} z \le \pi$

$\therefore 0 \le 2\cos^{-1} z \le 2\pi$

$\therefore 0 \le y \le 2\pi$ (D)

A	12
B	1
C	5
D	141

4.



$y^2 = f(x)$ is only defined if $f(x) \ge 0$

$\therefore$ A and D don't work.

B is $y = \sqrt{f(x)}$

C is $y = \pm\sqrt{f(x)} \Rightarrow y^2 = f(x)$ (C)

A	8
B	5
C	140
D	6

5. $\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}||\underset{\sim}{v}| \cos\theta$

$= 3 . 3 . \cos\frac{\pi}{3}$

$= \frac{9}{2}$ (C)

A	0
B	1
C	150
D	8

6. ${}^{20}C_4 = 4845$

(A)

A	132
B	1
C	25
D	1

7. The diagram indicates that $\frac{dy}{dx} = 0$ if $x = 0$

$\therefore$ A, B, D do not guarantee this

Also for C $\frac{dy}{dx} = 0$ if $y = 1$

$$\frac{dy}{dx} = xy - x$$
$$= x(y-1)$$

C is the only diagram meeting these requirements.

(C)

A	0
B	7
C	145
D	7

8. Volume $= \pi \int_1^2 x^2 \, dy$

$$y = e^{\frac{x}{2}}$$
$$\therefore \frac{x}{2} = \ln y$$
$$\therefore x = 2 \ln y$$

$$\therefore \text{Volume} = \pi \int_1^2 (2\ln y)^2 \, dy$$
$$= 4\pi \int_1^2 (\ln y)^2 \, dy$$

(D)

A	4
B	14
C	0
D	141

9. Projection of $\underset{\sim}{p}$ onto $\underset{\sim}{q}$

$$= \frac{\underset{\sim}{p} \cdot \underset{\sim}{q}}{\underset{\sim}{q} \cdot \underset{\sim}{q}} \underset{\sim}{q}$$

$$= \frac{\begin{pmatrix} 4 \\ -3 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix}}{\begin{pmatrix} -1 \\ 2 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix}} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$= \frac{-4-6}{1+4} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$= -2 \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

(B)

A	5
B	111
C	13
D	30

10. $\int \frac{x}{\sqrt{9-x^2}} \, dx$ Let $u = 9 - x^2$, $du = -2x \, dx$

$$= -\frac{1}{2} \int \frac{-2x \, dx}{\sqrt{9-x^2}}$$

$$= -\frac{1}{2} \int \frac{du}{u^{\frac{1}{2}}}$$

$$= -\frac{1}{2} \times 2u^{\frac{1}{2}} + c$$

$$= -u^{\frac{1}{2}} + c$$

$$= -\sqrt{9-x^2} + c$$

(A)

A	125
B	17
C	13
D	4

Question 11 (15 marks)

(a) Six chairs are to be painted. Two of the chairs are to be painted black. **1**
The other four are to be painted blue, green, yellow and grey.
The chairs are then arranged in a row.
How many possible arrangements are there for the colours of the chairs?

There are 6 chairs but 2 of them are indistinguishable from each other.

The total arrangements can be represented thusly: $\frac{6!}{2!} = 360$

NOTE: This was generally well done. Of the answers that were wrong, the most common mistake was to treat the 2 black chairs as a single thing to be arranged.
This solution does not include the all the variations where the two black chairs have other coloured chairs between them.

(b) (i) Find $\dfrac{dy}{dx}$ if $y = xe^{-x}$. **2**

Using the Product Rule, let $y = uv$, where $u = x$ and $v = e^{-x}$.

Then $y' = u'v + v'u$, where $u' = 1$ and $v' = -e^{-x}$.

Substituting these gives $y' = 1 \times e^{-x} + x \times -e^{-x}$

The above simplifies to $y' = e^{-x}(1-x)$.

(ii) Hence find $\displaystyle\int_0^1 xe^{-x}\,dx$. **3**

Solution 1:

From part (i) we know that $\displaystyle\int_0^1 e^{-x}(1-x)\,dx = \left[xe^{-x}\right]_0^1$.

Expand the integrand: $\displaystyle\int_0^1 e^{-x} - xe^{-x}\,dx = \left[xe^{-x}\right]_0^1$, which results in

$\displaystyle\int_0^1 e^{-x}\,dx - \int_0^1 xe^{-x}\,dx = \left[xe^{-x}\right]_0^1$, and rearranging this to make the subject the integral that we are trying to find: $\displaystyle\int_0^1 xe^{-x}\,dx = \int_0^1 e^{-x}\,dx - \left[xe^{-x}\right]_0^1$

Finally $\displaystyle\int_0^1 xe^{-x}\,dx = \left[-e^{-x}\right]_0^1 - \left[xe^{-x}\right]_0^1$ or $-\left[e^{-x} + xe^{-x}\right]_0^1$ *

Evaluating the RHS gives $-e^{-1} - 1 \times e^{-1} - (-e^{0} - 0 \times e^{0}) = 1 - \dfrac{2}{e}$

Solution 2:

$$\int_0^1 xe^{-x}\,dx = \int_0^1 e^{-x} - e^{-x} + xe^{-x}\,dx$$

$$= \int_0^1 e^{-x} - \frac{d}{dx}\left(xe^{-x}\right)dx$$

$$= -\left[e^{-x} + xe^{-x}\right]_0^1$$

NOTE: This was generally well done. The odd student treated the question like an indefinite integral or did not correctly do the plug in the boundaries at step (*).

Quite a number of students did not use integral notation correctly; that is, they couldn't be bothered writing *dx* at the end of the integral.

This is important because it identifies to readers what the integrand is so there is no confusion with other terms in an expression that are not part of the integrand.

(c) Solve $\dfrac{3}{x(2x-1)} \geq 1$? **3**

Solution 1:

The critical points for this graph are $x = 0$ and $\dfrac{1}{2}$ - these are asymptotes.

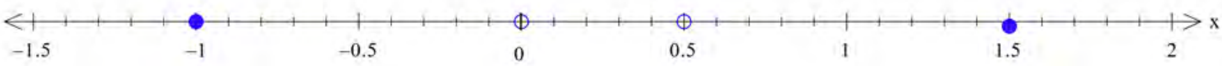
Solve the equality: $\dfrac{3}{x(2x-1)} = 1 \Rightarrow 0 = 2x^2 - x - 3$, where $x \neq 0, \dfrac{1}{2}$.

We get $(2x-3)(x+1) = 0$, where $x \neq 0, \dfrac{1}{2}$. Solutions are $x = -1, \dfrac{3}{2}$.

(*) Since there are asymptotes at zero and half, we need to check numbers between the critical values and see if they satisfy the original inequality.

These ranges are: $\left(-\infty, -1\right], \left[-1, 0\right), \left(0, \dfrac{1}{2}\right), \left(\dfrac{1}{2}, \dfrac{3}{2}\right], \left[\dfrac{3}{2}, \infty\right)$.

Visually:



The numbers we will use are: $-2, -\dfrac{1}{2}, \dfrac{1}{4}$, 1 and 2 which respectively relate to the regions above.

Substituting these values into $\dfrac{3}{x(2x-1)} - 1 \geq 0$ gives: $-\dfrac{7}{10}, 2, -25, 2, -\dfrac{1}{2}$ respectively.

Therefore the only regions that the inequality holds are: $\left[-1, 0\right), \left(\dfrac{1}{2}, \dfrac{3}{2}\right]$ OR $-1 \leq x < 0$ and $\dfrac{1}{2} < x \leq \dfrac{3}{2}$.

Solution 2:

Multiply BOTH sides by the square of the denominator and solve the resulting inequality.

This resulting inequality is: $3x(2x-1) \geq x^2(2x-1)^2$, where $x \neq 0, \dfrac{1}{2}$.

Take all terms to one side and factorise if possible:

$$0 \geq x^2(2x-1)^2 - 3x(2x-1)$$
$$\therefore 0 \geq x(2x-1)\left(x(2x-1) - 3\right)$$
$$\therefore 0 \geq x(2x-1)\left(2x^2 - x - 3\right)$$
$$\therefore 0 \geq x(2x-1)(2x-3)(x+1)$$

The rest of this solution follows Solution 1 from (*), but noting that $x \neq 0, \dfrac{1}{2}$.

(c) (continued)

Solution 3:

The critical points for this graph are $x = 0$ and $\frac{1}{2}$ - these are asymptotes.

Case 1: $x < 0$

When $x < 0$, we can multiply through by the denominator (which is always positive since $x < 0$ and $2x - 1 < 0$) and we get $3 \geq x(2x-1)$. Taking all terms to one side and solving the resulting quadratic gives $-1 \leq x \leq \frac{3}{2}$, but it is only valid for $x < 0$, thus $-1 \leq x < 0$.

Case 2: $0 < x < \frac{1}{2}$

When $0 < x < \frac{1}{2}$, we can multiply through by the denominator but this time the inequality sign needs to change (since the denominator will be negative - $x > 0$ and $2x - 1 < 0$) and we get $3 \leq x(2x-1)$.

Solving this quadratic after taking all terms to one side gives $x < -1$ and $x > \frac{3}{2}$.

Neither of these are within the domain of $\left(0, \frac{1}{2}\right)$.

Hence there is no solution in this region.

Case 3: $x > \frac{1}{2}$

When $x > \frac{1}{2}$, the denominator is once again positive and we may use the same quadratic and solutions from Case 1 and see if part of that solution lies in the stated range for Case 3.

One can see that $\frac{1}{2} < x \leq \frac{3}{2}$ is a solution in this case.

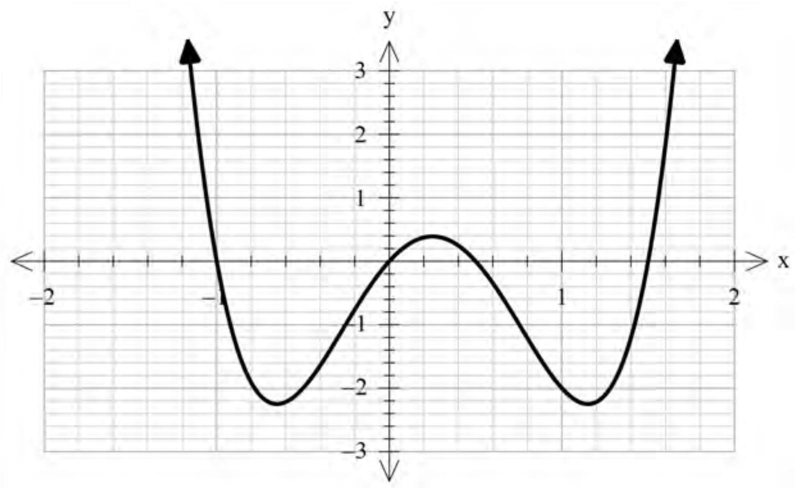
Solution is $-1 \leq x < 0$ and $\frac{1}{2} < x \leq \frac{3}{2}$ OR $\left[-1, 0\right) \cup \left(\frac{1}{2}, \frac{3}{2}\right]$.

(c) (continued)

NOTE: Rather than check specific values, students often relied on a visual representation of the graph.

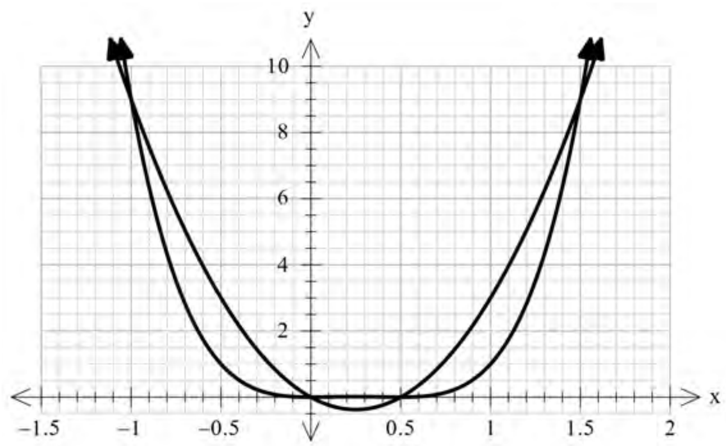
Some were able to graph the curve $y = \dfrac{3}{x(2x-1)}$ and identify the regions where the function was greater than 1.

Others realised the shape of the quartic and using its zeroes drew a close approximation which was good enough to identify the regions that were above
(or below as the case may be) the *x*-axis.



The graph above shows the expression $x(x+1)(2x-1)(2x-3)$ and the inequality should mean students are looking for the intervals where the curve is below the *x*-axis.

Some students compared the inequality $3x(2x-1) \geq x^2(2x-1)^2$.



This is not a bad idea. However, most students who did this failed to recognise that these graphs intersect at $x = -1$ and $\dfrac{3}{2}$.

The main mistake by most students was that they failed to state the intervals correctly at the end. They simply forgot that there must be an asymptote at $x = 0$ and $\dfrac{1}{2}$.

(d) The position vectors for the points P, Q, R and S are **p**, **q**, **r** and **s** respectively – shown below in column vector notation.

$$\mathbf{p} = \begin{pmatrix} -8 \\ -8 \end{pmatrix},\ \mathbf{q} = \begin{pmatrix} 3 \\ 6 \end{pmatrix},\ \mathbf{r} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \text{ and } \mathbf{s} = \begin{pmatrix} -5 \\ 7 \end{pmatrix}$$

Find

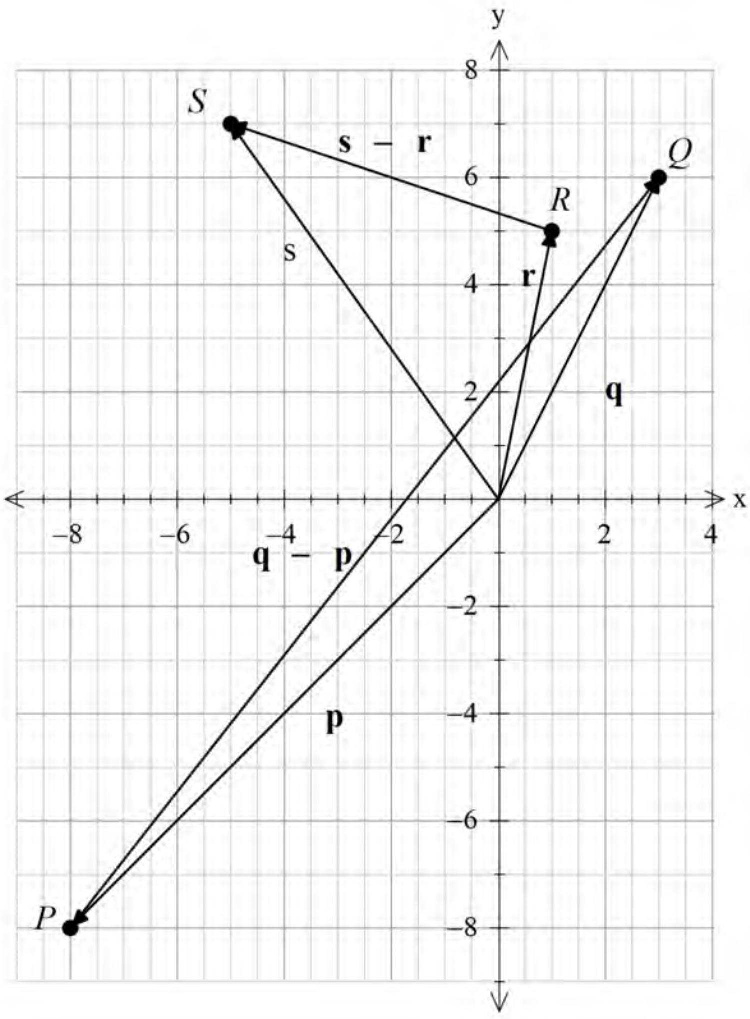
(i) $\overrightarrow{PQ}$ **1**

$\overrightarrow{PQ} = \mathbf{q} - \mathbf{p}$ which is: $\begin{pmatrix} 3 \\ 6 \end{pmatrix} - \begin{pmatrix} -8 \\ -8 \end{pmatrix} = \begin{pmatrix} 11 \\ 14 \end{pmatrix}$

(ii) $-\overrightarrow{PQ} - \overrightarrow{RS}$ **1**

$$\begin{aligned} -\overrightarrow{\mathrm{PQ}} - \overrightarrow{\mathrm{RS}} &= -(\mathbf{q} - \mathbf{p}) - (\mathbf{s} - \mathbf{r}) \\ &= \begin{pmatrix} -8 \\ -8 \end{pmatrix} - \begin{pmatrix} 3 \\ 6 \end{pmatrix} + \begin{pmatrix} 1 \\ 5 \end{pmatrix} - \begin{pmatrix} -5 \\ 7 \end{pmatrix} \\ &= \begin{pmatrix} -5 \\ -16 \end{pmatrix} \end{aligned}$$

NOTE: A visual representation of the points and position vectors is supplied below. Numerous arithmetic errors were made by students and some students thought that $\overrightarrow{OP} = \mathbf{p} + \mathbf{q}$.



(e) Find the coefficient of x in the expansion of $\left(2x-\frac{1}{2x}\right)^9$. **2**

Each term in the expansion above looks like: ${}^9C_r\left(2x\right)^{9-r}\left(-\frac{1}{2x}\right)^r$, where $r=\{0,1,...,9\}$.

By inspection, it can be seen that if $r=4$, we will get ${}^9C_4 2x=252x$ and the coefficient of x is 252.

But if it is not obvious then one can work it out by comparing terms to the one we want:

$$Ax = {}^9C_r(2x)^{9-r}\left(-\frac{1}{2x}\right)^r$$

$$= (-1)^r \times {}^9C_r \times 2^{9-r}2^{-r} \times x^{9-r}x^{-r}$$

$$= (-1)^r\, {}^9C_r 2^{9-2r} x^{9-2r}$$

Hence if we solve the powers for x, we get $9-2r=1$, and r is clearly 4.
Then we substitute this into the coefficient on the RHS and the result is as above.

NOTE: Most students worked out the correct value for r mathematically.
A small number obviously found it by inspection.
A few did not treat the negative sign appropriately or mixed up r and $9-r$ when finding the coefficient required.
Others left out the ${}^9C_r$ when stating the coefficient of x.

(f) Differentiate $y = e^x \arcsin(2x)$. **2**

Use Product rule: Let $y = uv$, where $u = e^x$ and $v = \arcsin(2x)$.

Note that arcsin(θ) is an alternative way of stating the function $\sin^{-1}(\theta)$.

Then $u' = e^x$ and $v' = \frac{d}{dx}(w) \times \frac{d}{dw}\arcsin(w)$, where $w = 2x$ (Chain rule).

So $v' = \frac{2}{\sqrt{1-(2x)^2}}$.

$y' = u'v + v'u$, hence $y' = e^x \times \arcsin(2x) + \frac{2}{\sqrt{1-4x^2}} \times e^x$.

Finally $y' = e^x \left(\arcsin(2x) + \frac{2}{\sqrt{1-4x^2}} \right)$.

NOTE: This was done very well.
Only a small handful of students did not get something very close to the correct result.
Using the reference sheet would have helped those who did not get this (unless of course brushing up on the Product rule is the remedy).

Those who got $1\frac{1}{2}$ rather than full marks will recognise easily what it was they left out.

Question 12

a) With 9 red, 9 blue, 9 yellow and 9 black socks, Marty still does not have 10 of a single colour.

Current total = 36

∴ The 37^{th} sock must mean he meets the required condition.

* Question done well, but still a large number not getting it correct.

b) Step 1 - Show for $n=1$

$4^1 + 14 = 18$

$= 6 \times 3$, where $3 \in \mathbb{Z}^+$

Step 2 - Assume for $n=k$

i.e. $4^k + 14 = 6P$, where $P \in \mathbb{Z}^+$

$4^k = 6P - 14$

Step 3 - Show for $n=k+1$

RTP: $4^{k+1} + 14 = 6Q$, where $Q \in \mathbb{Z}^+$

LHS $= 4^{k+1} + 14 = 4 \times 4^k + 14$

$= 4(6P-14) + 14$, by the inductive step

$= 24P - 42$

$= 6(4P-7) = 6Q$, where $Q = 4P-7 \in \mathbb{Z}^+$.

$=$ RHS

∴ true for $n=k+1$

Step 4 - The statement is true by Mathematical Induction.

* Students completed question well.

* Students who didn't specify that P (or equivalent variable) is an integer were penalised $\frac{1}{2}$.

* Students must set up steps 1 & 2 for 1 mark, clearly show inductive step for 2^{nd} mark, and have a full solution for 3 marks.

c) $(n+2)! = 72 \times n!$

$(n+2)(n+1) \times n! = 72 \times n!$

$(n+2)(n+1) = 72$

$n^2 + 3n + 2 = 70$

$n^2 + 3n - 70 = 0$

$(n-7)(n+10) = 0$

$n = -10, 7$, but $n \geq 0$, so

$n = 7$.

* Students completed question well.
* Some students made simple arithmetic errors that led to quadratics with irrational roots.
* Students who solved $(n+2)(n+1) = 72$ by inspection received full marks.
* Students who did not exclude -10 as a solution penalised $\frac{1}{2}$ mark.

d) i) ${}^{14}C_2 \times {}^{13}C_2 = 7098$

* Most students received full marks
* Students who calculated ${}^{14}P_2 \times {}^{13}P_2$ or ${}^{14}C_2 + {}^{13}C_2$ received 1 mark

ii) $13 \times 12 = 156$

* Well done
* Students who showed 14×13 received $\frac{1}{2}$ mark.
* Students who had ${}^{14}C_2 + {}^{13}C_2$ in (i) were not penalised for having $13 + 12 = 25$ as their answer.

e) i) $4x^2 + 9y^2 = 36$

$\frac{d}{dx}(4x^2) + \frac{d}{dx}(9y^2) = \frac{d}{dx}(36)$

$\frac{d}{dx}(4x^2) + \frac{dy}{dx} \times \frac{d}{dy}(9y^2) = \frac{d}{dx}(36)$

$8x + \frac{dy}{dx} \times 18y = 0$

$18y \frac{dy}{dx} = -8x$

$\frac{dy}{dx} = \frac{-8x}{18y} = \frac{-4x}{9y}$ as required

* Question completed well.
* Students who started at $\frac{dy}{dx} = -\frac{4x}{9y}$ to show $4x^2 + 9y^2 = 36$ were not penalised, but should take care.

ii) Let $x = 2$

$$4 \times 2^2 + 9y^2 = 36$$
$$9y^2 = 20$$
$$y^2 = \frac{20}{9}$$
$$y = \sqrt{\frac{20}{9}} = \frac{2\sqrt{5}}{3} \quad \text{(since } y > 0\text{)}$$

$$\frac{dy}{dx} = \frac{-4 \times 2}{9 \times \frac{2\sqrt{5}}{3}} \quad \left(\text{or } \frac{-8}{6\sqrt{5}},\ -\frac{4}{3\sqrt{5}},\ -\frac{4\sqrt{5}}{15},\ -0.596 \text{ or equivalent}\right)$$

* Question was done well.
* Some students found y, but did not correctly substitute into $\frac{dy}{dx}$
* Some students found correct expressions, but attempted to simplify incorrectly. Subsequent errors were ignored.

f) i) $Q(x) = Ax^3 + Bx^2 + Cx + D$

$Q'(x) = 3Ax^2 + 2Bx + C$

Considering the quadratic equation when $Q'(x) = 0$,

$$3Ax^2 + 2Bx + C = 0$$
$$\Delta = b^2 - 4ac$$
$$= (2B)^2 - 4 \times 3A \times C$$
$$= 4B^2 - 12AC$$
$$= 4(B^2 - 3AC)$$
$$< 0$$

This shows that there are no real x for which $Q'(x) = 0$, so $Q(x)$ has no stationary points.

$\therefore$ $Q(x)$ is either always increasing or always decreasing.

Since $Q(x)$ is a cubic, it has exactly 1 zero.

* Very few students received full marks.
* Many students determined that there are no real x where $Q'(x) = 0$, but made the weaker claim that there are no multiple roots. This received $1\frac{1}{2}$ marks.
* Students who found $Q'(x)$ received $\frac{1}{2}$, and those who found the discriminant of $Q'(x)$ received 1.
* Some students spoke of $B^2 - 3AC$ as the discriminant of $Q(x)$, or attempted to use it as discriminant of $Q'(x)$ without justification.
* Students who used roots α, β and γ were all unsuccessful at making progress answering the question.

ii) If $B^2 - 3AC = 0$,

the discriminant of the quadratic equation $Q'(x) = 0$ is zero.

$\therefore$ The equation has a double root at $x = \frac{-b}{2a}$

$$= \frac{-2B}{6A}$$

$$= \frac{-B}{3A}$$

$\therefore Q\left(\frac{-B}{3A}\right) = Q'\left(\frac{-B}{3A}\right) = Q''\left(\frac{-B}{3A}\right) = 0$

$\therefore$ Multiplicity is 3.

Alternate method:

$Q'(x) = 3Ax^2 + 2Bx + C$

$Q'\left(\frac{-B}{3A}\right) = 3A \times \left(\frac{-B}{3A}\right)^2 + 2B \times \left(\frac{-B}{3A}\right) + C$

$= \frac{3AC - B^2}{3A} = 0$, since $B^2 - 3AC = 0$

$Q''(x) = 6Ax + 2B$

$Q''\left(\frac{-B}{3A}\right) = 6A \times \frac{-B}{3A} + 2B$

$= 0$

$\therefore Q\left(\frac{-B}{3A}\right) = Q'\left(\frac{-B}{3A}\right) = Q''\left(\frac{-B}{3A}\right) = 0$

* Question was done poorly.
* Mark was awarded for a bald answer.

Question 13

(a) (i) $K = \frac{21000}{7+3e^{-\frac{t}{3}}} = 21000(7+3e^{-\frac{t}{3}})^{-1}$ Let $u = 7 + e^{-\frac{t}{3}}$, so $\frac{du}{dt} = -e^{-\frac{t}{3}}$

$$K = 21000u^{-1}, \quad \frac{dK}{du} = -\frac{21000}{u^2}$$

$$\frac{dK}{dt} = \frac{dK}{du} \times \frac{du}{dt} \quad \text{(chain rule)}$$

$$= \frac{21000}{e^{-\frac{t}{3}}(7+3e^{-\frac{t}{3}})^2}$$

1 mark for differentiating

$$\text{RHS} = \frac{1}{3}\left(1 - \frac{K}{3000}\right)K$$

$$= \frac{1}{3}\left(1 - \frac{7}{7+3e^{-\frac{t}{3}}}\right)\left(\frac{21000}{7+3e^{-\frac{t}{3}}}\right)$$

$$= \left(\frac{7+3e^{-\frac{t}{3}}-7}{7+3e^{-\frac{t}{3}}}\right)\left(\frac{7000}{7+3e^{-\frac{t}{3}}}\right)$$

$$= \frac{21000e^{-\frac{t}{3}}}{(7+3e^{-\frac{t}{3}})^2}$$

$$= \frac{dK}{dt}$$

= LHS.

1 mark for matching differential equation with derivative

Notes:

- Differentiate K first (the alternative of integrating is very difficult).
- Next use LHS and RHS so we know what we are referring to.

(ii) Sub. in $t = 0$: $K = \frac{21000}{7+3e^0} = 2100$

1 mark

(iii) Find the limit as $t \to \infty$: $K = \frac{21000}{7+3e^{-\frac{t}{3}}}$ $\quad (\lim_{t\to\infty} e^{-t} = 0)$

$$= \frac{21000}{7}$$

$$= 3000$$

1 mark

Note: A clever alternative was to find when the differential equation is zero. But explain how this finds the eventual population by the graph.

(iv) At $t = 0$: $\frac{dK}{dt} = \frac{21000}{e^0(7+3e^0)^2} = \frac{21000}{100} = 210$ kangaroos/year

½ mark

So $\frac{210}{2100} \times 100\% = 10\%$ annual rate.

½ mark

Notes:

- Today is $t = 0$.
- Read the question about answer needed as a percentage.

(b) Solve $\frac{\sin x + \cos x}{\sqrt{2}} = \frac{\sqrt{3}}{2}$ for $0° \le x \le 360°$.

Method 1: (squaring)

$\sin x + \cos x = \frac{\sqrt{6}}{2} > 1$, so $\sin x > 0$, $\cos x > 0$

i.e. $0° < x < 90°$

1 mark for this or restricting values later

Squaring: $(\sin x + \cos x)^2 = \frac{3}{2}$

$\sin^2 x + 2\sin x \cos x + \cos^2 x = \frac{3}{2}$

$2\sin x \cos x = \frac{3}{2} - 1 = \frac{1}{2}$ **1 mark**

$\sin 2x = \frac{1}{2}$

$2x = \sin^{-1}\frac{1}{2}$ for . $0° < 2x < 180°$

$= 30°, 150°$

$\therefore x = 15°, 75°$ **1 mark**

Method 2: (Auxiliary angle)

$R\sin(x + \alpha) = R\cos\alpha \sin x + R\sin\alpha \cos x$ (compound angle expansion)

$= \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x$ (given)

So $R\cos\alpha = \frac{1}{\sqrt{2}}, R\sin\alpha = \frac{1}{\sqrt{2}}$ (matching coefficients) **1 mark**

Squaring and adding: $R^2(\cos^2\alpha + \sin^2\alpha) = \frac{1}{2} + \frac{1}{2}$

$R^2 = 1$

$R = 1$

Then $\cos\alpha = \frac{1}{\sqrt{2}}$, $\sin\alpha = \frac{1}{\sqrt{2}}$, so $\alpha = 45°$ (or $\frac{\pi}{4}$) **1 mark**

So $\sin(x + 45°) = \frac{\sqrt{3}}{2}$

$x + 45° = \sin^{-1}(\frac{\sqrt{3}}{2}), 180° - \sin^{-1}(\frac{\sqrt{3}}{2})$

$= 60°, 120°$

$\therefore x = 15°, 75°$ **1 mark**

(b) (continued)

Method 3: (t-formula)

Let $t = \tan\frac{x}{2}$, so $\sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}$

Then $\frac{1}{\sqrt{2}}\left(\frac{2t+(1-t^2)}{1+t^2}\right) = \frac{\sqrt{3}}{2}$ **1 mark**

$4t + 2 - 2t^2 = \sqrt{6}(1 + t^2)$

$(\sqrt{6} + 2)t^2 - 4t + (\sqrt{6} - 2) = 0$

$t = \frac{4\pm\sqrt{16-4(6-4)}}{2(\sqrt{6}+2)} = \frac{2\pm\sqrt{2}}{\sqrt{6}+2}$ **1 mark for solving**

So $\tan\frac{x}{2} = \frac{2\pm\sqrt{2}}{\sqrt{6}+2}$

$x = 2\tan^{-1}\left(\frac{2\pm\sqrt{2}}{\sqrt{6}+2}\right)$

$= 15°,\ 75°$ **1 mark**

Notes:

- Remember $\sin x = \cos(90° - x)$, so there are two solutions.
- Define t before you use it in the t formula method.
- Check your solutions are valid.
- Check whether 180° is a solution

(c) Find the domains and main points for the graph:

arcsin(–1) = $-\frac{\pi}{2}$ When $3x+1 = -1$, $x = -\frac{2}{3}$

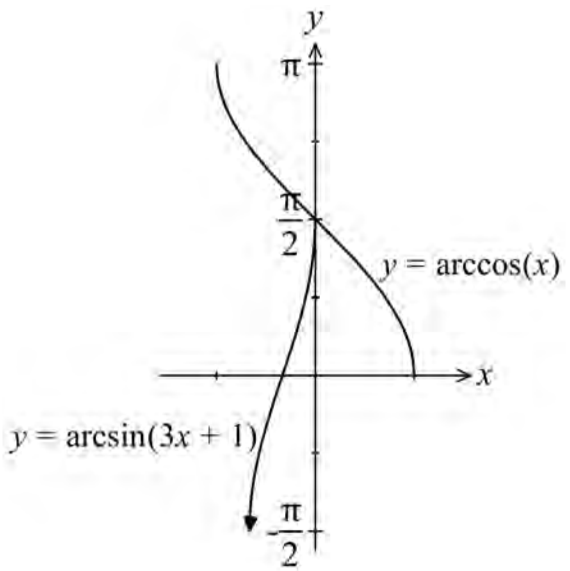
arcsin(0) = 0 When $3x+1 = 0$, $x = -\frac{1}{3}$

arcsin(1) = $\frac{\pi}{2}$ When $3x+1 = 1$, $x = 0$

arccos(–1) = π

arcsin(0) = $\frac{\pi}{2}$

arccos(1) = 0



2 marks

1 mark

$\therefore x = 0.$

Notes:

- The solution is for *x*, NOT for *y*.
- The suggested method (graphing) is usually the best, so try it.

(d) Find $\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}} dx$, using $x = \sin\theta$.

$\frac{dx}{d\theta} = \cos\theta \;\; so \; dx = \cos\theta \, d\theta$

Top limit: $\frac{1}{2} = \sin\theta$ so $\theta = \frac{\pi}{6}$

1 mark for differentials and limits

Subbing: $\int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \cos\theta \, d\theta = \int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{\cos\theta} \cos\theta \, d\theta$

$= \int_0^{\frac{\pi}{6}} \sin^2\theta \, d\theta$

1 mark for simplifying to here

$= \frac{1}{2}\int_0^{\frac{\pi}{6}} (1 - \cos\theta \, d\theta$ (since $\cos 2\theta = 1 - 2\sin^2\theta$)

$= \frac{1}{2}[\theta - \frac{sin 2\theta}{2}]_0^{\frac{\pi}{6}}$

1 mark for integration

$= \frac{1}{2}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{2\times 2}\right) - \frac{1}{2}(0 - 0)$

1 mark for solution

$= \frac{\pi}{12} - \frac{\sqrt{3}}{8}$

Note:

- Substituting in the variables, differentials and limits are easy marks. You should be doing them quickly.
- Don't waste time factorising your results (unless it is easy and clean).

QUESTION 14 SOLUTIONS

(a) A golfer hits a golf ball from a point O with speed V m/s at an angle θ above the horizontal, where $0° < \theta < 90°$. The ball just passes over a 2.25 m high tree after 1.5 seconds. The tree is 60 metres away on level ground from the point from which the ball was hit. The acceleration due to gravity is 10 m/s^2.

The position vector, $\underset{\sim}{r}$, at any time, t seconds, is given by

$$\underset{\sim}{r}(t) = Vt\cos\theta\,\underset{\sim}{i} + \left(Vt\sin\theta - 5t^2\right)\underset{\sim}{j}$$

(i) What is the initial angle, θ, of projection of the golf ball, to the nearest minute? **3**

$$\underset{\sim}{r}(1.5) = 1.5V\cos\theta\underset{\sim}{i} + \left(1.5V\sin\theta - 11.25\right)\underset{\sim}{j}$$
$$= 60\underset{\sim}{i} + 2.25\underset{\sim}{j}$$

Comparing components:

$$\begin{cases} 1.5V\cos\theta = 60 \\ 1.5V\sin\theta - 11.25 = 2.25 \end{cases}$$

$$\therefore \begin{array}{ll} V\cos\theta = 40 & -(1) \\ V\sin\theta = 9 & -(2) \end{array}$$

$$(2)\div(1) \Rightarrow \quad \tan\theta = \frac{9}{40}$$
$$\therefore \theta \doteqdot 12°41'$$

Comment:

This was generally done well by most students, though some students were not satisfied with a straightforward problem and decided to challenge themselves by introducing the trajectory equation. Some did not live up to this challenge.

Some students still feel the need to waste time and prove the results that are given to them.
Some students made simple mistakes by not simplifying as soon as possible at each step.

(ii) What is the initial speed (V m/s) of the golf ball, correct to the nearest whole number? **1**

From part (i), consider $(1)^2 + (2)^2$:

$$(V\cos\theta)^2 + (V\sin\theta)^2 = 40^2 + 9^2$$
$$\therefore V^2(\cos^2\theta + \sin^2\theta) = 41^2$$

$\therefore$ the initial speed is 41 m/s

Comment:

Some students used their rounded value of the angle in part (i). Fortunately for most, it rounded to the exact value.
Students are reminded that unless they are told to use a previous value, they should always deal with the 'exact' value and then round if needed.

(a)	(iii)	How far from the base of the tree does the ball strike the ground? Leave your answer correct to the nearest metre.	**1**

$$\underset{\sim}{r}(t) = t \times \underbrace{V\cos\theta}_{40}\,\underset{\sim}{i} + \left(t \times \underbrace{V\sin\theta}_{9} - 5t^2\right)\underset{\sim}{j}$$
$$= 40t\,\underset{\sim}{i} + \left(9t - 5t^2\right)\underset{\sim}{j}$$

The ball will land when $9t - 5t^2 = 0 \Rightarrow t = \dfrac{9}{5} = 1.8$

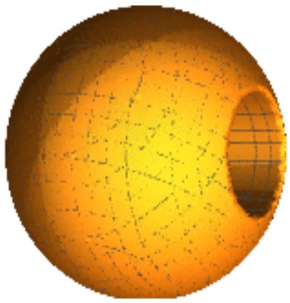
So the ball lands $40 \times 1.8 = 72$ m from the start which is 12 m from the tree.

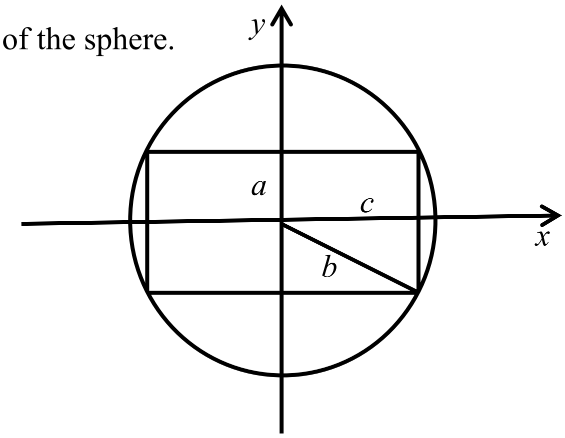
Comment:

Many students made this harder by not recognising previous results i.e. $V\cos\theta = 40$ and $V\sin\theta = 9$.

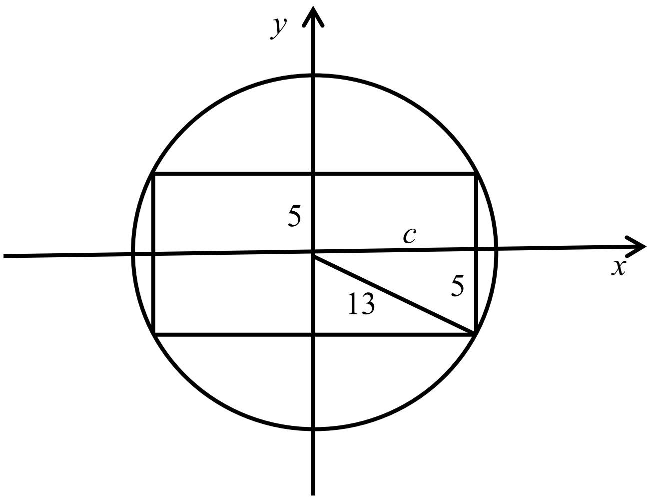
Sadly, many students did not read the wording of the question and just provided the answer here as 72 m. This time they were not penalised.

(b) To make a macramé bead, a cylindrical hole of radius 5 mm is bored through a sphere of radius 13 mm.
The axis of the cylinder coincides with a diameter of the sphere.





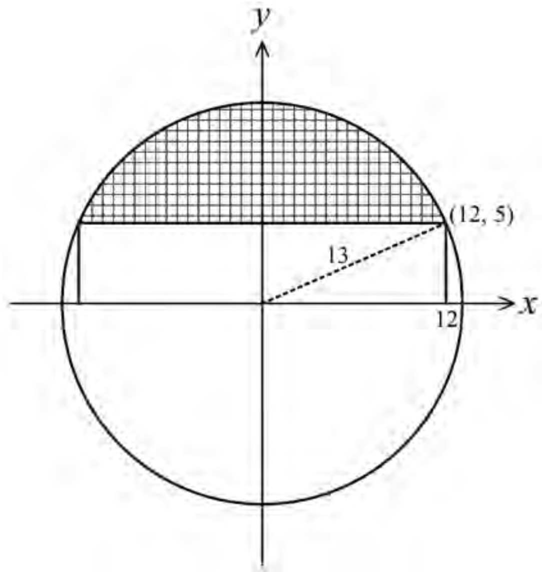
(i) State the values of *a, b,* and *c* in the diagram. **2**



$a = 5$, $b = 13$ and $c = 12$ (Pythagoras' Thm)

Comment:
For what should have been the most straightforward question in the paper, students managed to make this difficult. And not just a few students.

(b)	(ii)	Find the volume of the bead, by using a rotation about the x-axis.	**4**



The volume of the bead is obtained by rotating the area between the circle $x^2 + y^2 = 169$ and the line $y = 5$ for $-12 \le x \le 12$ i.e. the shaded area.

$$V = \pi \int_{-12}^{12} \left(y^2 - 5^2\right) dx$$

$$= 2\pi \int_{0}^{12} \left(169 - x^2 - 25\right) dx$$

$$= 2\pi \int \left(144 - x^2\right) dx$$

$$= 2\pi \left[144x - \frac{1}{3}x^3\right]_0^{12}$$

$$= 2\pi \times 1152 \text{ cu}$$

$$= 2304\pi \text{ cu}$$

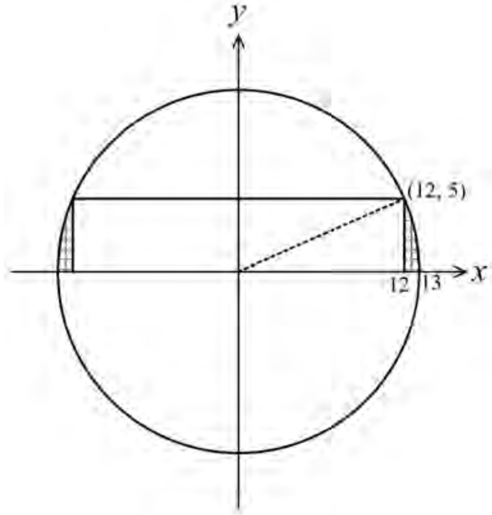
Note: this is the same volume as that of a sphere radius 12.

Alternative method over the page

(b) (ii) (continued)

Find the volume of the cap that is formed when the shaded area (below) is rotated about the x-axis.

$V_{\text{bead}} = V_{\text{sphere}} - V_{\text{cylinder}} - V_{\text{cap}}$



$$
\begin{aligned}
V_{\text{cap}} &= 2 \times \pi \int_{12}^{13} (169 - x^2)\, dx \\
&= 2\pi \left[169x - \frac{1}{3}x^3 \right]_{12}^{13} \\
&= 2\pi \left[\left(169 \times 13 - \frac{1}{3} \times 13^3 \right) - \left(169 \times 12 - \frac{1}{3} \times 12^3 \right) \right] \\
&= 2\pi \times 12\frac{2}{3} \\
&= \frac{76\pi}{3}
\end{aligned}
$$

$$
\begin{aligned}
V_{\text{bead}} &= V_{\text{sphere}} - V_{\text{cylinder}} - V_{\text{cap}} \\
&= \frac{4}{3}\pi \times 13^3 - \pi \times 5^2 \times 24 - \frac{76\pi}{3} \\
&= 2304\pi
\end{aligned}
$$

Comment:

The most common strategy used was to say that the volume of the bead was the difference of the volume of the sphere and the volume of the cylinder. Students who went down this path could only get a maximum of 1 mark. Why? Apart from being wrong, this could be done very simply using these formulae: $V_{\text{sphere}} = \frac{4}{3}\pi r^3$ and $V_{\text{cylinder}} = \pi R^2 H$.

Also, many students who did choose the correct strategy still elected to use calculus to work out these volumes when needed rather than use the above formulae. This is not helping with time management.

(c) (i) By considering $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_nx^n$, where $n \in \mathbb{Z}^+$, show that **2**

$$n2^{n-1} = {}^nC_1 + 2\,{}^nC_2 + 3\,{}^nC_3 + \ldots + n\,{}^nC_n.$$

Differentiate both sides wrt x: $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_nx^n$

$$\therefore n(1+x)^{n-1} = {}^nC_1 + 2\,{}^nC_2x + \ldots + n\,{}^nC_nx^{n-1}$$

Substitute $x = 1$: $n(1+1)^{n-1} = {}^nC_1 + 2\,{}^nC_2 + \ldots + n\,{}^nC_n$

$$\therefore n2^{n-1} = {}^nC_1 + 2\,{}^nC_2 + \ldots + n\,{}^nC_n$$

Comment: It was quite surprising that many students did not know how to do this.

(c) (ii) By considering the coefficient of x^n in the expansion of **3**

$$\left(1+x+x^2+\ldots+x^n\right)^2\left(1+x\right)^n,$$

find the number of combinations of n letters out of $3n$ letters of which n letters are 'a', n letters are 'b' and the rest of the letters are unlike.

$$\left(1+x+x^2+\ldots+x^n\right)^2\left(1+x\right)^n=\left(1+x+x^2+\ldots+x^n\right)^2\left({}^nC_0+{}^nC_1x+{}^nC_2x^2+\ldots+{}^nC_nx^n\right)$$

We need the coefficient of x^n from:

$$\left({}^nC_0+{}^nC_1x+{}^nC_2x^2+\ldots+{}^nC_nx^n\right)\times\left(1+x+x^2+\ldots+x^n\right)\times\left(1+x+x^2+\ldots+x^n\right)$$

Pick ${}^nC_0$: Need coefficient of x^n from $\left(1+x+x^2+\ldots+x^n\right)\left(1+x+x^2+\ldots+x^n\right)$

$\therefore$ coefficient of $1\times x^n+x\times x^{n-1}+\ldots+x^n\times 1$ i.e. $n+1$

Note: This is equivalent to the number of ways of picking a total of n letters consisting of 'a's and 'b's only.

Pick ${}^nC_1x$: Need coefficient of x^{n-1} from $\left(1+x+x^2+\ldots+x^n\right)\left(1+x+x^2+\ldots+x^n\right)$

$\therefore$ coefficient of $1\times x^{n-1}+x\times x^{n-2}+\ldots+x^{n-1}\times 1$ i.e. n

Note: This is equivalent to the number of ways of picking a total of $(n-1)$ letters consisting of 'a's and 'b's and 1 letter from the distinct group.

Pick ${}^nC_2x^2$: Need coefficient of x^{n-2} from $\left(1+x+x^2+\ldots+x^n\right)\left(1+x+x^2+\ldots+x^n\right)$

$\therefore$ coefficient of $1\times x^{n-2}+x\times x^{n-3}+\ldots+x^{n-2}\times 1$ i.e. $n-1$

Note: This is equivalent to the number of ways of picking a total of $(n-2)$ letters consisting of 'a's and 'b's and 2 letters from the distinct group.

$\vdots$

Pick ${}^nC_nx^n$: Need coefficient of x^0 from $\left(1+x+x^2+\ldots+x^n\right)\left(1+x+x^2+\ldots+x^n\right)$

$\therefore$ coefficient of 1×1 i.e. 1.

Note: This is equivalent to the number of ways of picking all n letters from the distinct group.

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(c) (ii) So the coefficient of x^n is ${}^nC_0(n+1)+{}^nC_1n+{}^nC_2(n-1)+...+{}^nC_n\times 1$.

$$
\begin{aligned}
{}^nC_0(n+1)+{}^nC_1n+{}^nC_2(n-1)+...+{}^nC_n\times 1 &= {}^nC_0(n+1)+{}^nC_1\big((n+1)-1\big) \\
&\quad +{}^nC_2\big((n+1)-2\big)+...+{}^nC_n\times\big((n+1)-n\big) \\
&= (n+1)\left({}^nC_0+{}^nC_1+...+{}^nC_n\right) \\
&\quad -\left({}^nC_1+2\,{}^nC_2+...+n\,{}^nC_n\right)
\end{aligned}
$$

Consider $\left(1+x\right)^n={}^nC_0+{}^nC_1x+{}^nC_2x^2+...+{}^nC_nx^n$ and substitute $x = 1$.

$\therefore {}^nC_0+{}^nC_1+...+{}^nC_n=2^n$

So the coefficient of x^n:

$$
\begin{aligned}
\therefore {}^nC_0(n+1)+{}^nC_1n+{}^nC_2(n-1)+...+{}^nC_n\times 1 &= (n+1)2^n-n2^{n-1} \\
&= 2(n+1)2^{n-1}-n2^{n-1} \\
&= 2^{n-1}\left(2n+2-n\right) \\
&= (n+2)2^{n-2}
\end{aligned}
$$

The number of combinations of n letters out of $3n$ letters of which n letters are 'a', n letters are 'b' and the rest of the letters are unlike is $(n+2)2^{n-2}$

Comment: Many students either forgot about the combinations of letters or just re-wrote the question with addressing the issue.

Many students tried to start by recognising a geometric series. This did not lead anywhere. Also, some students tried to say that picking k 'a's or 'b's is $\binom{n}{k}$. Given that the 'a's are identical this also led to a quick end.

End of solutions