SYDNEY GRAMMAR SCHOOL

CANDIDATE NUMBER

2020 Trial Examination

Form VI Mathematics Extension 1

Monday 17th August 2020

General Instructions

- Reading time — 10 minutes
- Working time — 2 hours
- Attempt all questions.
- Write using black pen.
- Calculators approved by NESA may be used.
- A loose reference sheet is provided separate to this paper.

Total Marks: 70

Section I (10 marks) Questions 1–10
- This section is multiple-choice. Each question is worth 1 mark.
- Record your answers on the provided answer sheet.

Section II (60 marks) Questions 11–14
- Relevant mathematical reasoning and calculations are required.
- Start each question in a new booklet.

Collection

- If you use multiple booklets for a question, place them inside the first booklet for the question.
- Arrange your solutions in order.
- Write your candidate number on this page, on the start of the separate section and on the multiple choice sheet.
- Place everything inside this question booklet.

Checklist
- Reference sheet
- Multiple-choice answer sheet
- 4 booklets per boy
- Candidature: 128 pupils

Writer: RCF

Section I

Questions in this section are multiple-choice.
Choose a single best answer for each question and record it on the provided answer sheet.

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1. A projectile has an initial velocity vector $\underset{\sim}{v} = \begin{bmatrix} 2\sqrt{3} \\ 2 \end{bmatrix}$. Which of the following is the correct statement of its initial speed and angle of projection from the horizontal?

   (A) $4\,\text{m/s}$ at 30°

   (B) $\sqrt{10}\,\text{m/s}$ at 30°

   (C) $4\,\text{m/s}$ at 60°

   (D) $\sqrt{10}\,\text{m/s}$ at 60°

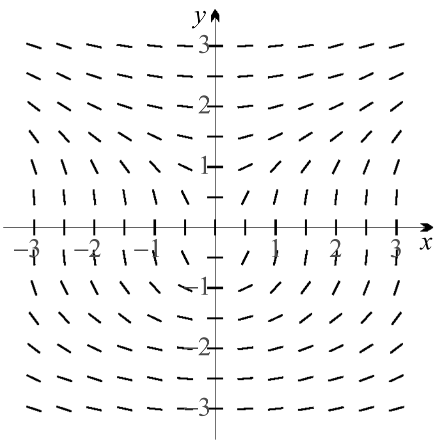
2. What is the coefficient of the x^3 term in the expansion of $(2-3x)^8$?

   (A) 56

   (B) -1512

   (C) 1512

   (D) $-48\,384$

3. 

   Which of the following differential equations could produce the slope field shown above?

   (A) $\dfrac{dy}{dx} = \dfrac{x}{y^2}$

   (B) $\dfrac{dy}{dx} = xy$

   (C) $\dfrac{dy}{dx} = \dfrac{x}{y}$

   (D) $\dfrac{dy}{dx} = x^2 y$

4. Given $\underset{\sim}{a} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$, which of the following represents $\text{proj}_{\underset{\sim}{b}}\,\underset{\sim}{a}$?

   (A) $\begin{bmatrix} 40 \\ -20 \end{bmatrix}$

   (B) $\begin{bmatrix} 20 \\ -10 \end{bmatrix}$

   (C) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

   (D) $\begin{bmatrix} \frac{4}{5} \\ -\frac{2}{5} \end{bmatrix}$

5. Which expression is equivalent to $1 - \cos 2x$?

   (A) $\sin^2 x - \cos^2 x$

   (B) $2\left(\sin^2 x + 1\right)$

   (C) $2 \sin^2 x$

   (D) $2\left(1 + \cos^2 x\right)$

6. A school mathematics department consists of 5 male and 5 female teachers. How many different exam-writing committees comprising three teachers could be formed if there must be at least one teacher of each sex on the committee?

   (A) 50

   (B) 100

   (C) 120

   (D) 600

7. Which of the following is a correct primitive of $\tan x$?

   (A) $\sec^2 x + c$

   (B) $\ln |A \sin x|$

   (C) $\dfrac{1}{1 + x^2} + c$

   (D) $-\ln |A \cos x|$

8. Which of the following is equivalent to $\displaystyle\int_0^{\frac{\pi}{3}} \cos^5 x \sin x \, dx$, after applying the substitution $u = \cos x$?

   (A) $\displaystyle\int_0^{\frac{\pi}{3}} u^5 \, du$

   (B) $\displaystyle -\int_{\frac{1}{2}}^{1} u^5 \, du$

   (C) $\displaystyle -\int_0^{\frac{\pi}{3}} u^5 \, du$

   (D) $\displaystyle\int_{\frac{1}{2}}^{1} u^5 \, du$

9. What is the value of $\cos^{-1}(\sin\alpha)$, where $\frac{\pi}{2} < \alpha < \pi$?

   (A) $\pi - \alpha$

   (B) $\frac{\pi}{2} - \alpha$

   (C) $\alpha - \frac{\pi}{2}$

   (D) $\alpha + \frac{\pi}{2}$

10. In trying to solve $\frac{x-2}{x-3} < \frac{4}{\sqrt{x-2}}$ over its natural domain in the set of real numbers, three students produce the following inequalities.

    Student I $\quad (x-2)\sqrt{x-2} < 4(x-3)$

    Student II $\quad \sqrt{(x-2)^3}(x-3) < 4(x-3)^2$

    Student III $\quad (x-2)^2(x-3) < 4(x-3)^2\sqrt{x-2}$

    Which students are still on track to obtain the correct solution set?

    (A) Just student II

    (B) Student III only

    (C) Both students II and III but not student I

    (D) All three students

End of Section I

The paper continues in the next section

CANDIDATE NUMBER

Section II

This section consists of long-answer questions.
Marks may be awarded for reasoning and calculations.
Marks may be lost for poor setting out or poor logic.
Start each question in a new booklet.

QUESTION ELEVEN (15 marks) Start a new answer booklet. **Marks**

(a) The polynomial $P(x) = x^3 - x^2 + kx - 4$ has a factor $(x - 1)$. Find the value of k. **1**

(b) Find the exact value of $\displaystyle\int_0^2 \frac{1}{4 + x^2}\, dx$. **2**

(c) Consider the curve $f(x) = 3\sin^{-1}\left(\dfrac{x}{2}\right)$.

(i) Sketch the curve, clearly indicating the coordinates of any intercepts with the axes and any endpoints. **2**

(ii) Find the exact gradient of the tangent to the curve at the point where $x = \frac{1}{2}$. **2**

(d) Use the substitution $u = 2x - 1$ to find $\displaystyle\int 4x\sqrt{2x - 1}\, dx$. **2**

(e) The equation $x^3 + bx^2 + 6x + d = 0$ has roots $1 + \sqrt{3}$, $1 - \sqrt{3}$ and 4. Use the sum and product of roots to find the integers b and d. **2**

(f) A particle moves in two dimensional space where $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are unit vectors in the x and y directions respectively. At time t seconds its displacement from the origin is given by $\underset{\sim}{r} = (6t - 4t^2)\,\underset{\sim}{i} + 2t\,\underset{\sim}{j}$ where all lengths are measured in metres.

(i) Write down the particle's velocity vector in component form. **1**

(ii) Find the speed of the particle when $t = 2$. **1**

(iii) Show that the equation of the path of the particle is $x = 3y - y^2$. **2**

QUESTION TWELVE (15 marks) Start a new answer booklet. **Marks**

(a) Solve the equation $\sin 2x = \sqrt{2} \sin x$, for $0 \leq x \leq 2\pi$. 3

(b) Solve the initial value problem where $\dfrac{dy}{dx} = 2xe^{-y}$ given $y(0) = \ln 4$. Express your answer with y as the subject. 3

(c) Points $P(4, -6)$, $Q(1, 2)$, $R(7, 5)$ form a triangle PQR in the Cartesian Plane.

(i) Find the vectors $\overrightarrow{QP}$ and $\overrightarrow{QR}$, representing two sides of this triangle. Give your answer in component form. 1

(ii) Use the dot product to find angle PQR. Give your answer correct to the nearest degree. 2

(d) Use mathematical induction to prove that, for positive integers n: 3

$$1 + 3 + 9 + \cdots + 3^{n-1} = \frac{1}{2}\left(3^n - 1\right).$$

(e) Write $\cos^2 2x$ in terms of $\cos 4x$ and hence evaluate $\displaystyle\int_0^{\frac{\pi}{6}} \cos^2 2x \, dx$. 3

QUESTION THIRTEEN (15 marks) Start a new answer booklet. **Marks**

(a) (i) Express $\sqrt{3}\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. 2

(ii) Hence, or otherwise, solve $\sqrt{3}\cos\theta - \sin\theta = -1$, for $0 \le \theta \le 2\pi$. 2

(b) The rate at which a cool object warms in air is proportional to the difference between its temperature T, in degrees Celsius, and the constant ambient temperature $A\,^\circ$C of the surrounding air. This rate can be expressed by the differential equation:

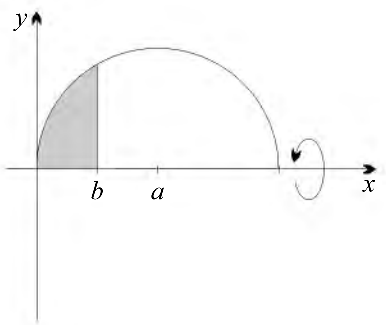
$$\frac{dT}{dt} = k(A - T)$$

where t is time in minutes and k is a positive constant. The solution of this differential equation is $T = A + Be^{-kt}$, where B is a constant. (You need NOT show this.)

A bottle of baby milk is at $4\,^\circ$C when it is removed from a refrigerator and placed on the kitchen bench where the room temperature is $22\,^\circ$C. Five minutes later it has warmed to $12\,^\circ$C. 3

Find the temperature of the milk after a further three minutes sitting on the bench. Give your answer correct to the nearest degree.

(c)

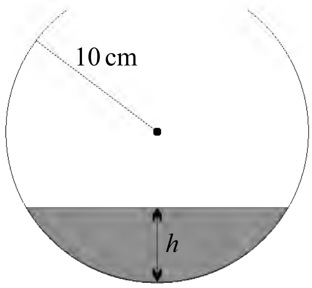


Consider the region enclosed by the upper semicircle $y = \sqrt{a^2 - (x - a)^2}$ and the vertical line $x = b$ where $0 < b < 2a$, shown shaded in the diagram above.

(i) A spherical cap is generated by rotating this region around the x-axis. Show that the volume V, in cubic units, of this solid is given by: 3

$$V = \frac{\pi b^2}{3}(3a - b).$$

(ii) 2



In the diagram above, a spherical vase of radius 10 cm is being filled with water at a constant rate 90 cm^3/min.

Let h cm be the depth of the water after t minutes. Find the rate at which the depth of the water is rising at the instant when the depth is 5 cm. Give your answer in terms of π.

(d) Use t formulae to solve $\sin x - 7\cos x = 5$, for $0 \le x \le 2\pi$. 3

QUESTION FOURTEEN (15 marks) Start a new answer booklet. **Marks**

(a) Let the function $f(x) = \sec x$ be defined for the restricted domain $0 < x < \frac{\pi}{2}$, so that its inverse $f^{-1}(x)$ is also a function.

(i) Show that $f^{-1}(x) = \cos^{-1}\left(\dfrac{1}{x}\right)$ and clearly state the domain of $f^{-1}(x)$. **2**

(ii) Hence find $\dfrac{d}{dx}\left(f^{-1}(x)\right)$ and describe the behaviour of the graph of $y = f^{-1}(x)$ as $x \to \infty$. **2**

(b) Data suggests that the number of cases of infection from a particular disease tends to fluctuate between two values over a period of approximately six months.

Let P be the number of cases after t months, where P is measured in <u>thousands</u>. Initially there are 1000 cases.

(i) Suppose that P is modelled by the equation: **2**

$$P = \frac{2}{2 - \sin t}.$$

Verify that P satisfies the differential equation $\dfrac{dP}{dt} = \dfrac{1}{2}P^2 \cos t$.

(ii) An alternative model is proposed with a different differential equation: **4**

$$\frac{dP}{dt} = \frac{1}{2}(2P^2 - P)\cos t.$$

Use the result that $\dfrac{1}{P(2P-1)} = \dfrac{2}{2P-1} - \dfrac{1}{P}$ to solve this differential equation, showing that:

$$P = \frac{1}{2 - e^{\frac{1}{2}\sin t}}$$

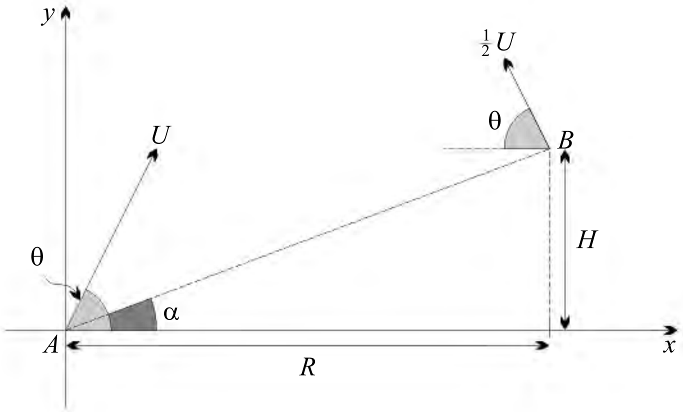
and that this new solution also satisfies the initial condition.

(iii) Find the greatest and least values of P predicted by both models for $t \geq 0$. Give your answers in exact form and then rounded to three decimal places to enable easy comparison between the two models. **2**

The question continues on the next page

QUESTION FOURTEEN (Continued)

(c) 3



In the diagram above, points A and B are separated by a horizontal distance R and point B is located H metres higher than A. Define the angle of inclination of B from A as α. Define the origin for both motions at point A and positive directions as right and up respectively.

At the same instant, identical projectiles are launched from each location directed towards each other. The projectile from A is fired with initial speed U m/s at an angle θ above the horizontal while the object launched from B has only half as much initial speed but the same angle of elevation above the horizontal.

Consequently, the equations of motion for the projectile from A, t seconds after launch, are as follows:

$$x_A = Ut\cos\theta \qquad y_A = Ut\sin\theta - \frac{1}{2}gt^2.$$

Similarly for the projectile from B:

$$x_B = R - \frac{Ut\cos\theta}{2} \qquad y_B = H + \frac{Ut\sin\theta}{2} - \frac{1}{2}gt^2.$$

[Do NOT prove these equations.]

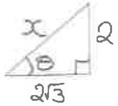
Given that the projectiles collide, show that this requires

$$\tan\alpha = \frac{1}{3}\tan\theta.$$

END OF PAPER

Extension 1, 2020 Trial.

① $\begin{pmatrix} 2\sqrt{3} \\ 2 \end{pmatrix}$



$x^2 = 2^2 + (2\sqrt{3})^2$
$= 4 + 12$
$= 16$
$x = 4$

$\tan\theta = \frac{2}{2\sqrt{3}}$
$= \frac{1}{\sqrt{3}}$
$\theta = 30°$

hence (A)

② $(2-3x)^8 = \ldots + {}^8C_5 \times 2^5 \times (-3x)^3 + \ldots$
$= \ldots + 56 \times 32 \times (-27)\, x^3 + \ldots$ hence (D)
$= \ldots - 48384x^3 + \ldots$

③ $x=0$ $\frac{dy}{dx}=0$ $y=0$ Gradients vertical hence not (B) or (D)

1^{st} quad $\frac{dy}{dx}>0$ 2^{nd} quad $\frac{dy}{dx}<0$ 3^{rd} quad $\frac{dy}{dx}<0$ hence not (C)

hence (A)

④ $\text{proj}_{\underset{\sim}{b}}\underset{\sim}{a} = \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\underset{\sim}{b}\cdot\underset{\sim}{b}}\,\hat{\underset{\sim}{b}}$

$= \frac{20-10}{4^2+(-2)^2}\begin{bmatrix} 4 \\ -2 \end{bmatrix}$

$= \frac{10}{20}\begin{bmatrix} 4 \\ -2 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ hence (C)

⑤ $1 - \cos 2x = 1 - (\cos^2 x - \sin^2 x)$
$= \sin^2 x + 1 - \cos^2 x$ not (A)
$= \sin^2 x + 1 - (1 - \sin^2 x)$
$= 2\sin^2 x$ hence (C)

⑥ 2M1F ${}^5C_2 \times {}^5C_1$
2F1M ${}^5C_2 \times {}^5C_1$

At least one of each sex $2 \times {}^5C_2 \times {}^5C_1 = 2 \times \frac{5\times4}{2} \times 5$
$= 100$ hence (B)

⑦ $\int \tan x\, dx = -\int \frac{-\sin x}{\cos x}\, dx$
$= -\ln|\cos x| + C$ hence (D) NOT (B)
$= -\ln|A\cos x|$ if $C = -\ln A$

⑧ $\int_0^{\frac{\pi}{3}} \cos^5 x \sin x\, dx$

$u = \cos x$
$du = -\sin x\, dx$

x	0	$\frac{\pi}{3}$
u	1	$\frac{1}{2}$

$= \int_{u=1}^{\frac{1}{2}} u^5(-du)$

$= -\int_1^{\frac{1}{2}} u^5\, du$

$= \int_{\frac{1}{2}}^{1} u^5\, du$ (Reversing limits) hence (D)

⑨ $\cos^{-1}(\sin\alpha)$ $\frac{\pi}{2} < \alpha < \pi$ ie 2^{nd} quad

$\sin\alpha > 0$
$= \frac{a}{c}$
$\therefore \cos^{-1}(\sin\alpha) = \cos^{-1}\left(\frac{a}{c}\right)$
$= \angle ABC$

$\angle BAC = \pi - \alpha$
$\therefore \angle ABC = \frac{\pi}{2} - \angle BAC$
$= \frac{\pi}{2} - (\pi - \alpha)$
$= \alpha - \frac{\pi}{2}$
hence (C)

alternatively: use test case
eg $\cos^{-1}\left(\sin\frac{5\pi}{6}\right) = \cos^{-1}\left(\frac{1}{2}\right)$
$= \frac{\pi}{3}$

If $\alpha = \frac{5\pi}{6}$ $\pi - \alpha = \frac{\pi}{6}$ not (A)
$\frac{\pi}{2} - \alpha = -\frac{2\pi}{6}$ not (B)
$\alpha + \frac{\pi}{2} = \frac{8\pi}{6}$ not (D)

⑩ Domain $x \neq 3$ and $x > 2$
$\times\sqrt{x-2}$
$\times(x-3)^2$
Student I in error since sign of $x-3$ is indeterminate
Student II is correct
Student III is also correct, multiplication by extra factor of $\sqrt{x-2}$ on both sides is unnecessary but still gives a valid inequation
hence (C)

QU 11

a) $P(x) = x^3 - x^2 + kx - 4$

$P(1) = 0 \quad 1 - 1 + k - 4 = 0$

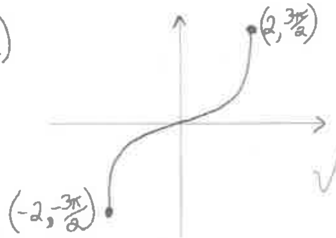
$k = 4$ ✓

b) $\int_0^2 \frac{1}{4+x^2}\,dx = \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^2$ ✓

$= \frac{1}{2}(\tan^{-1} 1 - \tan^{-1} 0)$ ✓

$= \frac{\pi}{8}$

c) $f(x) = 3\sin^{-1}\left(\frac{x}{2}\right)$



✓✓ correct shape + quadrants, correct endpts

$f'(x) = 3 \times \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}} \times \frac{1}{2}$

$= \frac{3}{2\sqrt{1-\frac{x^2}{4}}}$ ✓

$f'\left(\frac{1}{2}\right) = \frac{3}{2\sqrt{1-\frac{1}{16}}}$

$= \frac{3}{2\sqrt{\frac{15}{16}}}$

$= \frac{6}{\sqrt{15}}$

$= \frac{6\sqrt{15}}{15}$

$= \frac{2\sqrt{15}}{5}$ ✓

d) $\int 4x\sqrt{2x-1}\,dx$ Let $u = 2x-1$, $du = 2dx$

$= \int (u+1)\sqrt{u}\,du$ ✓

$= \int u^{3/2} + u^{1/2}\,du$ ✓

$= \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C$ ✓

$= \frac{2}{5}\sqrt{(2x-1)^5} + \frac{2}{3}\sqrt{(2x-1)^3} + C$

e) $x^3 + bx^2 + 6x + d = 0$ has roots $1+\sqrt{3}$, $1-\sqrt{3}$ and 4

$\sum\alpha = -\frac{b}{a} \therefore \quad 6 = -b \Rightarrow b = (-6)$

$\sum\alpha\beta\gamma = -\frac{d}{a} \therefore \quad 4(1+\sqrt{3})(1-\sqrt{3}) = -d$

$-8 = -d$

$d = 8$

Alternatively: $(x-4)(x-1-\sqrt{3})(x-1+\sqrt{3})$

$= (x-4)(x^2-2x-2)$

$x^3 - 6x^2 + 6x + 8$

f) $\underset{\sim}{r} = 6t - 4t^2\,\underset{\sim}{i} + 2t\,\underset{\sim}{j}$

(i) $\underset{\sim}{v} = \frac{dr}{dt} = (6-8t)\underset{\sim}{i} + 2\underset{\sim}{j}$ m/s ✓ Do not penalise units

(ii) $(\underset{\sim}{v})_{t=2} = -10\underset{\sim}{i} + 2\underset{\sim}{j}$

speed $= |\underset{\sim}{v}| = \sqrt{(-10)^2 + 2^2}$

$= \sqrt{104}$

$= 2\sqrt{26}$ m/s ✓ (Penalise units here)

(iii) $x = 6t - 4t^2$ ① $\quad y = 2t$ ②

from ② $t = \frac{y}{2}$

sub t into ①

$x = 6\left(\frac{y}{2}\right) - 4\left(\frac{y}{2}\right)^2$

$x = 3y - y^2$ ✓

QU12

a) $\sin 2x = \sqrt{2}\sin x \quad 0 \le x \le 2\pi$

$2\sin x\cos x - \sqrt{2}\sin x = 0$

$\sin x(2\cos x - \sqrt{2}) = 0$ ✓

$\sin x = 0$ OR $\cos x = \frac{1}{\sqrt{2}}$

$x = 0, \pi, 2\pi$ / OR $x = \frac{\pi}{4}, \frac{7\pi}{4}$ ✓

b) $\frac{dy}{dx} = 2xe^{-y}$ where $y(0) = \ln 4$

No constant soln $e^{-y} \neq 0$ ✓

Separate Variables $\int e^{y}\,dy = \int 2x\,dx$

$e^{y} = x^2 + C$ ✓

$y = \ln|x^2 + C|$

$y(0) = \ln 4 \quad \therefore \ln 4 = \ln|C|$

$C = 4$

$\therefore y = \ln|x^2 + 4|$ ✓

c) $P(4,-6)\ Q(1,2)\ R(7,5)$

$\vec{QP} = \underset{\sim}{p} - \underset{\sim}{q} = \begin{pmatrix} 4-1 \\ -6-2 \end{pmatrix} = \begin{pmatrix} 3 \\ -8 \end{pmatrix} = 3\underset{\sim}{i} - 8\underset{\sim}{j}$

$\vec{QR} = \underset{\sim}{r} - \underset{\sim}{q} = \begin{pmatrix} 7-1 \\ 5-2 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} = 6\underset{\sim}{i} + 3\underset{\sim}{j}$

✓ Both Vectors either column vector or Cartesian

$\vec{QP}\cdot\vec{QR} = 3\times 6 + (-8)\times 3 = 18 - 24 = (-6) = |QP||QR|\cos\angle PQR$

$|QP| = \sqrt{3^2 + 8^2} = \sqrt{73}$ $\quad |QR| = \sqrt{6^2 + 3^2} = \sqrt{45} = 3\sqrt{5}$ ✓

$\therefore \cos\angle PQR = \frac{-6}{3\sqrt{5}\sqrt{73}} = \frac{-2}{\sqrt{365}}$

$\angle PQR = \cos^{-1}\frac{-2}{\sqrt{365}}$

$\doteqdot 96°$ (nearest degree) ✓

d) $1 + 3 + 9 + \ldots + 3^{n-1} = \frac{1}{2}(3^n - 1)$

STEP A: Prove true for $n = 1$

LHS $= 1$ RHS $= \frac{1}{2}(3^1 - 1) = \frac{1}{2}\times 2 = 1$ $\quad \therefore$ True for $n = 1$

STEP B: Assume true for $n = k \quad k \in \mathbb{Z}^+$

ie $1 + 3 + 9 + \ldots + 3^{k-1} = \frac{1}{2}(3^k - 1)$

Prove true for $n = k+1$

ie $1 + 3 + 9 + \ldots + 3^{k-1} + 3^k = \frac{1}{2}(3^{k+1} - 1)$

✓ Both statements

LHS $= (1 + 3 + 9 + \ldots 3^{k-1}) + 3^k$

$= \frac{1}{2}(3^k - 1) + 3^k$ (By assumption) ✓

$= \frac{3}{2}\times 3^k - \frac{1}{2}$

$= \frac{1}{2}(3\times 3^k - 1)$

$= \frac{1}{2}(3^{k+1} - 1)$

$=$ RHS

$\therefore$ If true for $n = k$ also true for $n = k+1$

STEP C: By principle of Mathematical Induction conjecture true for all positive integer n.

✓ Well structured argument

e) $\cos^2 2x = \frac{1}{2}(1 + \cos 4x)$ ✓

$\int_{x=0}^{\pi/6} \cos^2 2x\,dx = \int_0^{\pi/6} \frac{1}{2} + \frac{1}{2}\cos 4x\,dx$

$= \left[\frac{x}{2} + \frac{1}{8}\sin 4x\right]_0^{\pi/6}$ ✓

$= \left(\frac{\pi}{12} + \frac{1}{8}\sin\frac{2\pi}{3}\right) - 0$

$= \frac{\pi}{12} + \frac{\sqrt{3}}{16}$ ✓

$= \frac{4\pi + 3\sqrt{3}}{48}$

QU 13

a) (i) $\sqrt{3}\cos\theta - \sin\theta = R\cos(\theta+\alpha)$
$R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

Equating coeffs $\cos\theta$, $R\cos\alpha = \sqrt{3}$ Ⓐ
$R\sin\alpha = 1$ Ⓑ

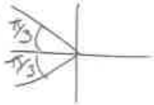
$Ⓐ^2 + Ⓑ^2$ $R^2(\cos^2\alpha + \sin^2\alpha) = 3+1$
$R^2 = 4$
$R = 2$ $(R>0)$ ✓

$\therefore \sin\alpha = \frac{1}{2}$ $\alpha = \frac{\pi}{6}$ ✓

ie $2\cos(\theta + \frac{\pi}{6})$

(ii) $2\cos(\theta+\frac{\pi}{6}) = -1$ $\qquad 0 \le \theta \le 2\pi$
$\cos(\theta+\frac{\pi}{6}) = -\frac{1}{2}$ $\qquad \frac{\pi}{6} \le \theta + \frac{\pi}{6} \le \frac{13\pi}{6}$ ✓

$\theta + \frac{\pi}{6} = \frac{2\pi}{3}, \frac{4\pi}{3}$
$\theta = \frac{\pi}{2}$ or $\frac{7\pi}{6}$ ✓



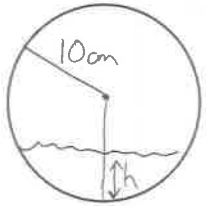
b) $T = A + Be^{-kt}$
$T = 22 + Be^{-kt}$ (room temp is 22°C)
$t=0$ $T=4$
$4 = 22 + Be^0$
$B = (-18)$
$\therefore T = 22 - 18e^{-kt}$ ✓
$t=5$ $T=12$
$12 = 22 - 18e^{-5k}$
$18e^{-5k} = 10$
$e^{-5k} = 5/9$
$e^{5k} = 9/5$
$5k = \ln(9/5)$
$k = \frac{1}{5}\ln\frac{9}{5}$ ✓ or equivalent $k = -\frac{1}{5}\ln(\frac{5}{9})$

$t=8$ $T=?$ $\quad T = 22 - 18e^{-\frac{8}{5}\ln\frac{9}{5}}$
$= 14.97...$ ✓

Temp will be approx 15°C

c) (i) $y = \sqrt{a^2 - (x-a)^2}$ (NB: a is a constant)
$V_x = \pi\int y^2\,dx$
$= \pi\int_{x=0}^{b} a^2 - (x-a)^2\,dx$ ✓ "SHOW"
$= \pi\int_0^b a^2 - x^2 + 2ax - a^2\,dx$ ✓
$= \pi\left[-\frac{x^3}{3} + ax^2\right]_0^b$ ✓ SHOW
$= \pi\left[\left(-\frac{b^3}{3} + ab^2\right) - (0)\right]$
$= \frac{\pi b^2}{3}(3a - b)$ $\quad h^2$

(ii) $a=10$ $\frac{dV}{dt} = 90\,cm^3/min$



Let depth of water be h.
$V = \frac{\pi h^2}{3}(30-h)$
$= 10\pi h^2 - \frac{\pi h^3}{3}$
$\frac{dV}{dh} = 20\pi h - \pi h^2$

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ ✓
$90 = (20\pi h - \pi h^2) \times \frac{dh}{dt}$ ✓
$\frac{dh}{dt} = \frac{90}{\pi h(20-h)}$
$\left(\frac{dh}{dt}\right)_{h=5} = \frac{90}{5\pi \times 15}$
$= \frac{6}{5\pi}$ cm/min ✓

d) $\sin x - 7\cos x = 5$, $0 \le x \le 2\pi$
Let $t = \tan\frac{x}{2}$ $\therefore \sin x = \frac{2t}{1+t^2}$ $\cos x = \frac{1-t^2}{1+t^2}$

$\frac{2t}{1+t^2} - \frac{7(1-t^2)}{1+t^2} = 5$

$2t + 7t^2 - 7 = 5 + 5t^2$ ✓
$2t^2 + 2t - 12 = 0$
$t^2 + t - 6 = 0$
$(t+3)(t-2) = 0$ ✓
$t = 2$ or (-3)
$\tan\frac{x}{2} = 2$ or (-3)
$\frac{x}{2} \doteqdot 1.107$ or $1.892...$
$x \doteqdot 2.21$ or 3.79 ✓ (2dp)

QU14

a) $f(x) = \sec x \quad 0 \leqslant x < \frac{\pi}{2}$

$1 \leqslant f(x)$

Inverse $x = \sec y$

$x = \frac{1}{\cos y}$

$\cos y = \frac{1}{x}$

$y = \cos^{-1}(\frac{1}{x})$ ✓ "SHOW"

$f^{-1}(x) = \cos^{-1}(\frac{1}{x})$

Range of $f(x)$ is domain of $f^{-1}(x)$ ✓

ie $x \geqslant 1$

ii) $\frac{d}{dx} f^{-1}(x) = \frac{d}{dx} \cos^{-1}(\frac{1}{x})$

$= -\frac{1}{\sqrt{1-(\frac{1}{x})^2}} \times (-x^{-2})$

$= \frac{\frac{1}{x^2}}{\sqrt{1-\frac{1}{x^2}}}$ $(\times x^2)$ $(\times x^2)$

$= \frac{1}{x\sqrt{x^2-1}}$ ✓

as $x \to \infty \quad \frac{dy}{dx} \to 0^+$ ✓

Gradient decreases to zero ie graph has horizontal asymptote ($y = \frac{\pi}{2}$)

b)(i) $P = \frac{2}{2-\sin t}$

$= 2(2-\sin t)^{-1}$

$\frac{dP}{dt} = -2(2-\sin t)^{-2} \times (-\cos t)$ ✓

$= \frac{2\cos t}{(2-\sin t)^2}$

$= \frac{4}{(2-\sin t)^2} \times \frac{\cos t}{2}$ ✓ "SHOW"

$= \frac{P^2 \cos t}{2}$ (as required)

(ii) $\frac{dP}{dt} = \frac{1}{2} P(2P-1)\cos t$

Constant Solns $P=0$ and $P=\frac{1}{2}$

$\int \frac{1}{P(2P-1)} dP = \int \frac{\cos t}{2} dt$ ✓

$\int \frac{-1}{P} + \frac{2}{2P-1} dP = \frac{\sin t}{2} + c$

$\ln|2P-1| - \ln|P| = \frac{1}{2}\sin t + c$ "SHOW"

$\ln\left|\frac{2P-1}{P}\right| = \frac{1}{2}\sin t + c$

$\left|\frac{2P-1}{P}\right| = e^{\frac{1}{2}\sin t} \times e^c$

$= Ae^{\frac{1}{2}\sin t}$ $(A>0)$

$\frac{2P-1}{P} = Ae^{\frac{1}{2}\sin t}$ ✓ $(A>0$ or $A<0)$

Then include constant soln $P=\frac{1}{2}$ ie $A=0$

$2P-1 = APe^{\frac{1}{2}\sin t}$

$P(2-Ae^{\frac{1}{2}\sin t}) = 1$ ✓

$P = \frac{1}{2-Ae^{\frac{1}{2}\sin t}}$

but $t=0$ $P=1$ $\therefore 1 = \frac{1}{2-Ae^0} \Rightarrow A=1$ ✓

$\therefore P = \frac{1}{2-e^{\frac{1}{2}\sin t}}$

(iii) $P = \frac{2}{2-\sin t}$

$-1 \leqslant \sin t \leqslant 1$

$3 \geqslant 2-\sin t \geqslant 1$

$\therefore \frac{2}{3} \leqslant P \leqslant \frac{2}{1}$ ✓ ie $0.667 \leqslant P \leqslant 2$

Alternate Model

$P = \frac{1}{2-e^{\frac{1}{2}\sin t}}$

$-\frac{1}{2} \leqslant \frac{1}{2}\sin t \leqslant \frac{1}{2}$

$e^{-\frac{1}{2}} \leqslant e^{\frac{1}{2}\sin t} \leqslant e^{\frac{1}{2}}$

$2-e^{\frac{1}{2}} \leqslant 2-e^{\frac{1}{2}\sin t} \leqslant 2-e^{-\frac{1}{2}}$

$\frac{1}{2-\sqrt{e}} \geqslant P \geqslant \frac{1}{2-\frac{1}{\sqrt{e}}}$ ✓

$\frac{1}{2-\frac{1}{\sqrt{e}}} \leqslant P \leqslant \frac{1}{2-\sqrt{e}}$

$0.718 \leqslant P \leqslant 2.847$

c) If particles collide $x_A = x_B$ at time T

$\therefore UT\cos\theta = R - \frac{UT\cos\theta}{2}$

$\frac{3UT\cos\theta}{2} = R$

$T = \frac{2R}{3U\cos\theta}$ ✓

Also $y_A = y_B$ at time T

$UT\sin\theta - \frac{1}{2}gT^2 = H + \frac{UT\sin\theta}{2} - \frac{1}{2}gT^2$ ✓

$H = \frac{UT\sin\theta}{2}$

$= \frac{U\sin\theta}{2} \times \frac{2R}{3U\cos\theta}$

$H = \frac{R}{3}\tan\theta$ ✓

but $\tan\alpha = \frac{H}{R}$

$\therefore \tan\alpha = \frac{1}{3}\tan\theta$ (as required)