

AMC 2023 Intermediate Solutions

by TY WEBB

1. $10 : 50 \text{ am} + \frac{40}{2} \text{ min} = 11 : 10 \text{ am} \dots C$

2. $360^\circ - 2 \times 90^\circ - 130^\circ = 50^\circ \dots E$

3. $\frac{2+3+4}{7+8+9} = \frac{9}{24} = \frac{3}{8} \dots C$

4. $\frac{100 \times 50}{25 \times 25} = 4 \times 2 = 8 \dots E$

5. $57 \times 953 = 54321 \dots C$

6. $PQ \times PT = 10 \times PT = 60 \text{ cm}^2 \therefore PT = 6 \text{ cm} \dots D$

7. Direct = x , scenic = $y \Rightarrow y = x + 5$ and $x + y = 35$

$$\therefore \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 35 \end{pmatrix} \therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 35 \end{pmatrix} = \begin{pmatrix} 15 \\ 20 \end{pmatrix} \therefore x = 15 \dots C$$

8. $((2^0)^2)^3 = 1^6 = 1 \dots A$

9. $\frac{0.05}{50} = 0.001 \dots D$

10. $5x = 180^\circ$ and $x + y = 90^\circ$

$$\therefore \begin{pmatrix} 5 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 180^\circ \\ 90^\circ \end{pmatrix} \therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 180^\circ \\ 90^\circ \end{pmatrix} = \begin{pmatrix} 36^\circ \\ 54^\circ \end{pmatrix} \therefore y = 54^\circ \dots C$$

11. $1 + 2 + 8, 1 + 3 + 7, 1 + 4 + 6, 2 + 3 + 6, 2 + 4 + 5 \therefore 5 \text{ ways} \dots C$

12. Two digit number is $xy \Rightarrow 10x + y + 10y + x = 11(x + y) \Rightarrow 11|xy + yx$

$\frac{55}{11} = 5, \frac{110}{11} = 10, \frac{132}{11} = 12$ and $\frac{154}{11} = 14$ but $\frac{186}{11} = 16\frac{10}{11} \dots E$

13. $\frac{\frac{1}{2} \times 2 \times 6 + 2 \times \frac{1}{2} \times 2 \times 4}{4 \times 6} = \frac{14}{24} = \frac{7}{12} \dots E$

14. $6 \times 5 + \frac{3}{1} - \frac{4}{2} = 31 \dots B$

15. Ages are respectively $x, x + y, x + 2y, x + 3y \Rightarrow x + x + y = 2x + y = 18$ and $x + 2y + x + 3y = 2x + 5y = 34$

$$\therefore \begin{pmatrix} 2 & 1 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 18 \\ 34 \end{pmatrix} \therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 18 \\ 34 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

$\therefore x + 3y = 7 + 3 \times 4 = 19 \dots D$

16. $\lceil \log_2 1000 \rceil = 10 \dots B$

17. $\frac{3}{2} \times \frac{1}{2} \times 15(x + 15) = 15^2 \therefore x + 15 = 20 \therefore x = 5 \dots A$

18. $x, y, n \in \mathbb{Z}^+$ and $2x + ny = 25 \Rightarrow 1 \leq ny = 25 - 2x \leq 23 \therefore 1 \leq n \leq 23$,
 $1 \leq y \leq 23, 1 \leq x = \frac{25-ny}{2} \leq 12 \therefore n = 2k - 1, k \in \{1, 2, 3 \dots 12\} \dots B$

19. Large square edge = x , small square edge = $y \Rightarrow 4x + 2y = 1.1 \times 4x$

$$\therefore y = 0.2x \text{ and } \frac{(0.2x)^2}{x^2} = 0.04 = 4\% \dots B$$

$$20. 7(4+x-9) = 4(4+x) \therefore 7x - 35 = 16 + 4x \therefore 3x = 51 \therefore x = 17 \dots E$$

$$21. \frac{\pi}{2}(1^2 + 3^2 + 4^2) = 13\pi \dots E$$

22. $a+3 = b-3 = 3c = \frac{d}{3}$ and so $3a+9=d$ and also $a+b+c+d=32$ hence

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -3 & 0 \\ 1 & 1 & 1 & 1 \\ 3 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -6 \\ -3 \\ 32 \\ -9 \end{pmatrix} \therefore \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -3 & 0 \\ 1 & 1 & 1 & 1 \\ 3 & 0 & 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} -6 \\ -3 \\ 32 \\ -9 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \\ 2 \\ 18 \end{pmatrix}$$

and $9+18=27 \dots D$

23. $2(4x+8x+8 \times 4) = 2(7y+6y+6 \times 7) \therefore 12x = 13y+10$ and $x, y \in \mathbb{Z}, x < 10, y < 10$
 $\therefore 12|13y+10$. Now $12x = 12(y+1) + y - 2$. So remainder when $13y+10$ is divided by 12 is $y-2 \pmod{12}$, so remainder $y-2=0 \Rightarrow y=2 \therefore x=\frac{13 \times 2+10}{12}=3$ and $x+y=3+2=5 \dots A$

24. Suppose the 3-digit number abc is such that it is divisible by $a+b+c$.

If $f(a, b, c) = \frac{100a+10b+c}{a+b+c}$ then $f(a+1, b, c) \geq f(a, b, c) > f(a, b+1, c) > f(a, b, c+1)$
 \therefore minimum value = $f(1, 9, 9) = 10\frac{9}{19} < f(1, 8, 9) = 10\frac{1}{2} < f(1, 9, 8) = 11 < f(a, b, c)$ for all other values of a, b, c . Hence the minimum integer value of $f(a, b, c)$ is 11 ... C

$$25. 2^2 + 2^2 - 1^2 : 1^2 :: 7 : 1 \dots C$$

$$26. 2, 3, 4, 5, 6|x-1 \text{ and } 7|x \therefore \exists k \in \mathbb{Z} : x = 7(10k+3)$$

$$7(10 \times 0 + 3) = 21 = 3 \times 7$$

$$7(10 \times 1 + 3) = 91 = 4 \times 22 + 3$$

$$7(10 \times 2 + 3) = 161 = 3 \times 53 + 2$$

$$7(10 \times 3 + 3) = 231 = 3 \times 77$$

So we reject $k = 0, 1, 2, 3$. However with $k = 4$ we have

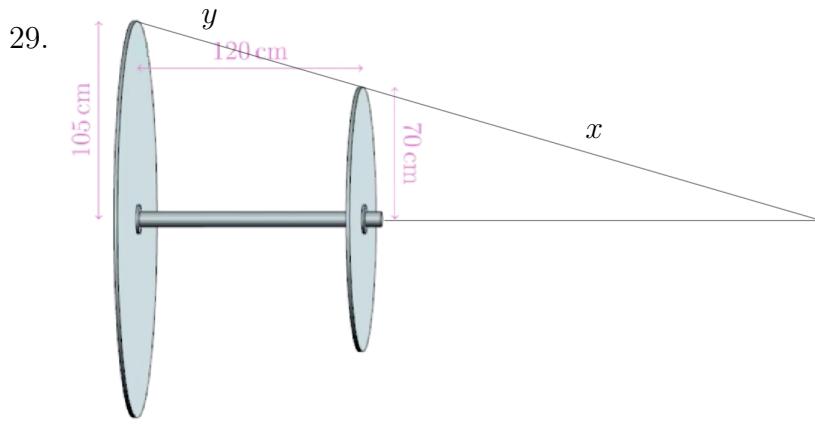
$$7(10 \times 4 + 3) = 301 = 2 \times 150 + 1 = 3 \times 100 + 1 = 4 \times 75 + 1 = 5 \times 60 + 1 = 6 \times 50 + 1 \text{ and so minimum number is 301.}$$

27. If $x = BC$, then $\tan \angle CGD = \frac{3}{4} = \frac{CD}{GC} = \frac{BC}{GC} = \frac{x}{GC} \therefore GC = \frac{4x}{3}$

$$\tan \angle AFB = \frac{4}{3} = \frac{BA}{FB} = \frac{CD}{FB} = \frac{x}{FB} \therefore FB = \frac{3x}{4}$$

$$GC + x + FB = x + \frac{4x}{3} + \frac{3x}{4} = \frac{37x}{12} = \sqrt{3^2 + 4^2} = 5 \therefore x = \frac{60}{37} \text{ and } 60 + 37 = 97$$

$$28. \frac{1000}{2} = 500 \text{ is even, then } \frac{500}{2} = 125 \text{ is odd } \therefore \text{number going is } \lfloor \frac{125}{2} \rfloor = 62$$

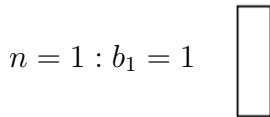


$$\frac{x+y}{x} = 1 + \frac{y}{x} = \frac{105}{70} = \frac{3}{2} \text{ and so } \frac{y}{x} = \frac{1}{2} \text{ so } x = 2y \text{ and } x + y = 2y + y = 3y.$$

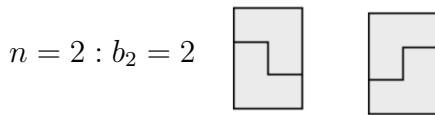
$$\text{Also, } y = \sqrt{35^2 + 120^2} = 125 \text{ and so } x + y = 3y = 375$$

30. Trominoe tilings of $3 \times n$ rectangles can be determined from basic shapes and then calculating the rest. Let b_n = number of basic shapes.

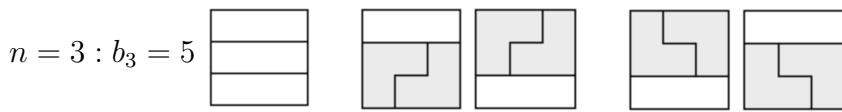
For the rest they can be determined recursively and inclusively as $a_0 = a_1 = b_1 = 1$ and for $n > 1, a_n = \sum_{i=1}^n a_{i-1} b_{n-i+1}$



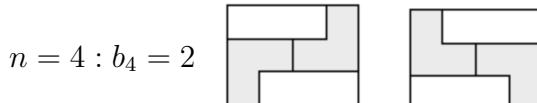
$$a_1 = 1$$



$$a_2 = a_0 b_2 + a_1 b_1 = 1 \times 2 + 1 \times 1 = 3$$



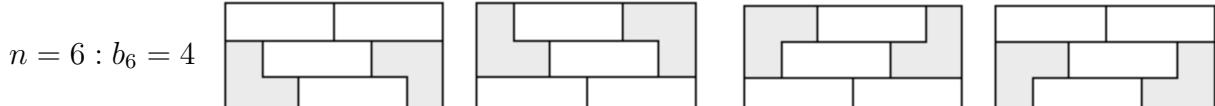
$$a_3 = a_0 b_3 + a_1 b_2 + a_2 b_1 = 1 \times 5 + 1 \times 2 + 3 \times 1 = 10$$



$$a_4 = a_0 b_4 + a_1 b_3 + a_2 b_2 + a_3 b_1 = 1 \times 2 + 1 \times 5 + 3 \times 2 + 10 \times 1 = 23$$



$$a_5 = a_0 b_5 + a_1 b_4 + a_2 b_3 + a_3 b_2 + a_4 b_1 = 1 \times 2 + 1 \times 2 + 3 \times 5 + 10 \times 2 + 23 \times 1 = 62$$



$$\begin{aligned} a_6 &= a_0 b_6 + a_1 b_5 + a_2 b_4 + a_3 b_3 + a_4 b_2 + a_5 b_1 \\ &= 1 \times 4 + 1 \times 2 + 3 \times 2 + 10 \times 5 + 23 \times 2 + 62 \times 1 \\ &= 170 \end{aligned}$$

Note for $n > 6$ this can be extended via a more efficient recursion

$a_n = a_{n-1} + 2a_{n-2} + 6a_{n-3} + a_{n-4} - a_{n-6}$ and so for $n = 0, 1, 2, 3, \dots$ we get a sequence $a_n = 1, 1, 3, 10, 23, 62, 170, 441, 1173, 3127, 8266, \dots$ as can be seen on the OEIS at <https://oeis.org/A134438>