AMC 2023 Senior Question 25 Alternative Solution

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Consider the following poem by Soddy and Gosset ([7] and [3]).

The Kiss Precise

FOR pairs of lips to kiss maybe
Involves no trigonometry.
'Tis not so when four circles kiss
Each one the other three.
To bring this off the four must be
As three in one or one in three.
If one in three, beyond a doubt
Each gets three kisses from without.
If three in one, then is that one
Thrice kissed internally.

Four circles to the kissing come.
The smaller are the benter.
The bend is just the inverse of
The distance from the centre.
Though their intrigue left Euclid dumb
There's now no need for rule of thumb.
Since zero bend's a dead straight line
And concave bends have minus sign,
The sum of the squares of all four bends
Is half the square of their sum.

To spy out spherical affairs
An oscular surveyor
Might find the task laborious,
The sphere is much the gayer,
And now besides the pair of pairs
A fifth sphere in the kissing shares.
Yet, signs and zero as before,
For each to kiss the other four
The square of the sum of all five bends
Is thrice the sum of their squares.

And let us not confine our cares
To simple circles, planes and spheres,
But rise to hyper flats and bends
Where kissing multiple appears.
In n-ic space the kissing pairs
Are hyperspheres, and Truth declares -
As $n + 2$ such osculate
Each with an $n + 1$ fold mate
The square of the sum of all the bends
Is n times the sum of their squares.

The Soddy-Gosset poem The Kiss Precise can be reformulated as follows, similar to an article by Lagarias, Mallows and Wilks ([4]).

Soddy-Gosset Theorem *Given an oriented Descartes configuration in* $\mathbb{R}^n$, *if we let* $c_1, \ldots c_{n+2}$ *be the oriented curvatures of the* $n+2$ *mutually tangent* n*-dimensional hyperspheres, then* $\left(\sum_{i=1}^{n+2} c_i\right)^2 = n \sum_{i=1}^{n+2} c_i^2$.

This has been proved by Pedoe in 1967 and Coxeter in 1968 ([5] and [2])

The 2023 AMC Senior paper question 25 ([1]) is as follows.

Three spheres of radius 2 sit on a flat surface touching one another. A smaller sphere sits on the same surface, in the middle and touching all three of the bigger spheres. What is its radius?

(A) $2\sqrt{3} - 2\sqrt{2}$ (B) $2\sqrt{\sqrt{3} - \sqrt{2}}$ (C) $\sqrt{3}$ (D) $\frac{3}{2}$ (E) $\frac{2}{3}$

Now we have the very efficient solution

By the Soddy-Gosset theorem with $n = 3$ and $c_1 = \frac{1}{r}, c_2 = c_3 = c_4 = \frac{1}{2}$ and $c_5 = 0$ we have that

$$\begin{aligned}
\left(\frac{1}{r} + 3 \times \frac{1}{2} + 0\right)^2 &= 3\left(\frac{1}{r^2} + 3\left(\frac{1}{2}\right)^2 + 0^2\right) \\
\therefore \frac{1}{r^2} + \frac{3}{r} + \frac{9}{4} &= \frac{3}{r^2} + \frac{9}{4} \\
\therefore 1 + 3r &= 3 \\
\therefore r &= \frac{2}{3}
\end{aligned}$$

Hence the answer is (E)

This contrasts starkly with the very complicated solutions that we see floating around on the internet such as the one on the ShareMyLesson youtube channel ([6])

References

[1] Australian Mathematics Competition 2023 Senior paper, Australian Mathematics Trust, 2023

[2] H. S. M. Coxeter, Loxodromic sequences of tangent spheres, Aequationes Mathematicae **1** (1968), 104–121.

[3] T. Gosset, The Kiss Precise. Nature (January 9, 1937), p. 62.

[4] J. C. Lagarias, C. L. Mallows and A. R. Wilks, Beyond the Descartes Circle Theorem, Amer. Math. Monthly **109** (2002), 338-361.

[5] D. Pedoe, On a theorem in geometry, Amer. Math. Monthly **74** (1967), 627–640.

[6] ShareMyLesson youtube video https://www.youtube.com/watch?v=Di-R_ReRL8Q

[7] F. Soddy, The Kiss Precise. Nature (June 20, 1936), p. 1021.