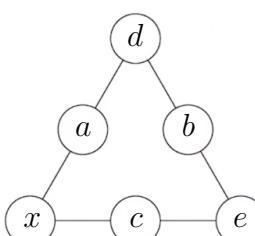


AMC 2023 Senior Solutions

by TY WEBB

1. $2023 - 3202 = -1179 \dots B$
2. $PQ \times PT = 10 \times PT = 60\text{cm}^2 \therefore PT = 6\text{cm} \dots D$
3. $57 \times 953 = 54321 \dots C$
4. $2^5 - 5^2 = 7 \dots E$
5. $\theta = 180^\circ - 110^\circ + 180^\circ - 150^\circ = 100^\circ \dots A$
6. $5^2 : 2 \times 3 \times 4 :: 25 : 24 \dots B$
7. $4 \times 5 \times 10 \times 15 = 3000 \therefore \prod_{i=1}^{18} i = \dots 000 \dots A$
8. $0.6x - 40 = 0.4x \therefore 0.2x = 40 \therefore x = 200\text{L} \dots D$
9. $\frac{4}{3}\pi r^3 = 100\text{cm}^3 \therefore r^3 = \frac{300}{4\pi} \approx \frac{300}{4} \times \frac{7}{22} = \frac{2100}{88} = 23\frac{19}{22} < 27 \therefore r < 3 \dots A$
10. $x + x + 2 + x + 4 = 3x + 6 = 9m + 3 \therefore x + 2 = 3m + 1 \therefore x + 4 = 3m + 3 \dots E$
11. $(\sqrt{24} + \sqrt{54})^2 = (2\sqrt{6} + 3\sqrt{6})^2 = (5\sqrt{6})^2 = 25 \times 6 = 150 \dots B$
12. $2^3 = 8 \dots B$
13. $\frac{a^{-1}-b^{-1}}{a^{-2}-b^{-2}} \times \frac{a^2b^2}{a^2b^2} = \frac{ab(b-a)}{b^2-a^2} = \frac{ab}{b+a} \dots C$
14. If total = x then after balls are removed,

$$\begin{aligned} \text{red balls} &= \frac{5x}{12} - 10 = \frac{5}{13}(x - 20) \\ \frac{5x-120}{12} &= \frac{5x-100}{13} \\ 65 - 1560 &= 60x - 1200 \\ 5x &= 360 \\ x &= 72 \dots C \end{aligned}$$
15. 

$$\begin{aligned} d &= a - x \text{ and } e = c - x \\ \therefore b &= a - x + c - x \\ 2x &= a + c - b \\ x &= \frac{a+c-b}{2} \dots D \end{aligned}$$
16. $f(g(x)) - g(f(x)) = 5 + 7 - x - (7 - (5 + x)) = 10 \dots D$
17. If the other sides are a, b then $a + b = 14 - 6 = 8$ and $a^2 + b^2 = 6^2$ so

$$ab = \frac{(a+b)^2 - 6^2}{2} = \frac{8^2 - 6^2}{2} = 14$$
 and so the area is $\frac{ab}{2} = \frac{14}{2} = 7\text{cm}^2 \dots A$

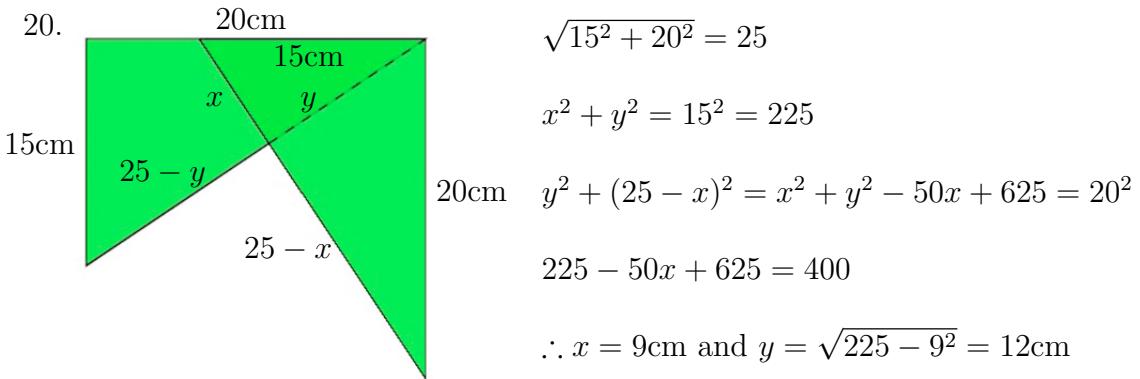
18. Just put a \checkmark for rational results in the table

\times	1	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
1	\checkmark	\checkmark				
$\frac{1}{2}$	\checkmark	\checkmark				
$\sqrt{3}$			\checkmark	\checkmark		\checkmark
$\frac{1}{\sqrt{3}}$			\checkmark	\checkmark		\checkmark
$\frac{1}{\sqrt{2}}$					\checkmark	
$\frac{\sqrt{3}}{2}$			\checkmark	\checkmark		\checkmark

$$\text{Probability} = \frac{14}{36} = \frac{7}{18} \dots E$$

19. $x \in \{2, 3, 4, 5, 6\}$ and $y \in \{1, 2, 3, 4, 5\}$ and most pairs (x, y) with $x > y$ satisfy the condition $x^2 + y^2 < 50$ but $(6, 4)$ and $(6, 5)$ do not satisfy the condition since $6^2 + 4^2 = 52 > 50$ and $6^2 + 5^2 = 61 > 50$ so number of pairs is $(1 + 2 + 3 + 4 + 5) - 2 = 13 \dots B$

20.



Now we have perimeter $= 15 + 20 + 20 + 25 - x + 25 - y = 105 - (x + y)$

$$\therefore \text{perimeter} = 105 - (9 + 12) = 105 - 21 = 84 \text{cm} \dots B$$

21. $a + 3 = b - 3 = 3c = \frac{d}{3}$ and so $3a + 9 = d$ and also $a + b + c + d = 32$ hence

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -3 & 0 \\ 1 & 1 & 1 & 1 \\ 3 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -6 \\ -3 \\ 32 \\ -9 \end{pmatrix} \therefore \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -3 & 0 \\ 1 & 1 & 1 & 1 \\ 3 & 0 & 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} -6 \\ -3 \\ 32 \\ -9 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \\ 2 \\ 18 \end{pmatrix}$$

and $9 + 18 = 27 \dots D$

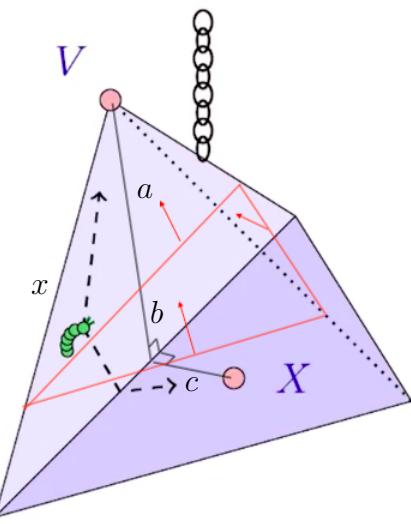
22. $a + b + c = 11.5$, $60(\frac{a}{3} + \frac{b}{4} + \frac{c}{5}) = 174$ and $60(\frac{c}{3} + \frac{b}{4} + \frac{a}{5}) = 186$ whereupon

$$\begin{pmatrix} 1 & 1 & 1 \\ 20 & 15 & 12 \\ 12 & 15 & 20 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 11.5 \\ 174 \\ 186 \end{pmatrix} \therefore \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 20 & 15 & 12 \\ 12 & 15 & 20 \end{pmatrix}^{-1} \begin{pmatrix} 11.5 \\ 174 \\ 186 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 4.5 \end{pmatrix}$$

so $b = 4 \dots A$

23. $S + P + 1 = x + y + xy + 1 = (x+1)(y+1)$ is a composite number but $88 + 1 = 89$ is prime... C

24.



$$\text{Total surface area} = 4 \times \frac{1}{2}x^2 \sin 60^\circ = \sqrt{3}x^2$$

Closer to V is above red lines and $a = b + c$

$$c = \frac{x}{2} \cot 60^\circ = \frac{x}{2\sqrt{3}}$$

$$\text{and } a + b + c = x \csc 60^\circ = \frac{2x}{\sqrt{3}}$$

$$a = \frac{a+b+c}{2} = \frac{x}{\sqrt{3}}$$

$$\text{and } b = a - c = \frac{x}{\sqrt{3}} - \frac{x}{2\sqrt{3}} = \frac{x}{2\sqrt{3}}$$

$$\text{Surface area above red lines} = 3 \times \frac{1}{2} \times 2a \cot 60^\circ \times a = \sqrt{3}a^2 = \sqrt{3} \times \left(\frac{x}{\sqrt{3}}\right)^2 = \frac{\sqrt{3}x^2}{3}$$

$$\text{Ratio of area above red lines to total} = \frac{\sqrt{3}x^2}{3} : \sqrt{3}x^2 = 1 : 3 \dots B$$

25. By the Soddy-Gosset Theorem, for curvatures k_1, \dots, k_{n+2} of $n+2$ touching n -dimensional hyperspheres, $(\sum_{i=1}^{n+2} k_i)^2 = n \sum_{i=1}^{n+2} k_i^2$ and so with $n=3$ and $k_1 = \frac{1}{r}, k_2 = k_3 = k_4 = \frac{1}{2}$ and $k_5 = 0$ we have that $(\frac{1}{r} + 3 \times \frac{1}{2})^2 = 3(\frac{1}{r^2} + 3(\frac{1}{2})^2)$
 $\therefore \frac{1}{r^2} + \frac{3}{r} + \frac{9}{4} = \frac{3}{r^2} + \frac{9}{4} \therefore 1 + 3r = 3 \therefore r = \frac{2}{3} \dots E$

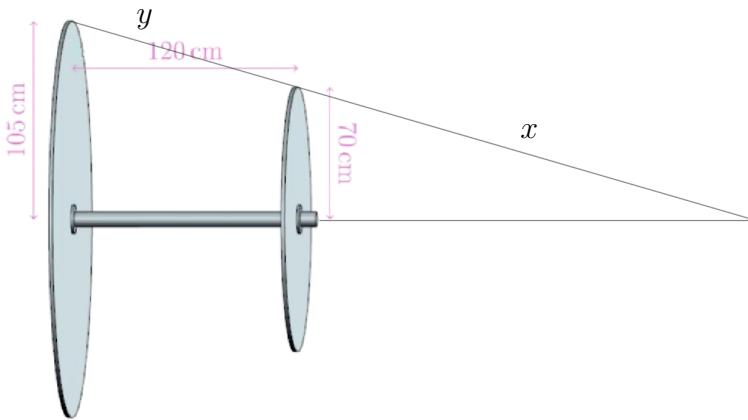
$$26. \sum_{i=1}^{13} (3k - 2) = \frac{13}{2}(1 + 37) = 247$$

$$\begin{aligned} 27. \frac{10^5 a + 10^4 b + 10^3 c + 10^2 a + 10b + c}{10^5 a + 10^4 b + 10^3 a + 10^2 b + 10a + b} &= \frac{1001(100a + 10b + c)}{10101(10a + b)} \\ &= \frac{11(100a + 10b + c)}{111(10a + b)} \\ &= \frac{11}{111} \left(10 + \frac{c}{10a + b}\right) \\ &= \frac{55}{54} \end{aligned}$$

$$\therefore \frac{c}{10a + b} = \frac{55}{54} \times \frac{111}{11} - 10 = \frac{5}{18} \therefore a = 1, b = 8, c = 5 \text{ and so}$$

$$100a + 10b + c = 100 + 80 + 5 = 185$$

28.



$$\frac{x+y}{x} = 1 + \frac{y}{x} = \frac{105}{70} = \frac{3}{2} \text{ and so } \frac{y}{x} = \frac{1}{2} \text{ so } x = 2y \text{ and } x + y = 2y + y = 3y.$$

$$\text{Also, } y = \sqrt{35^2 + 120^2} = 125 \text{ and so } x + y = 3y = 375$$

29. Where $F_1 = 2$ and $F_2 = 3$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$ it is $3! \times F_8 = 6 \times 55 = 330$

$$30. \sum_{i=0}^4 \frac{11}{i+1} \binom{9-i}{i} = 198$$