AMC 2025 Senior Solutions

by Ty Webb

1. $2^2 = 4 \therefore$ B

2. $\frac{132}{0.2} = 660 \therefore$ C

3. $(4^3)^2 - (2^3)^4 = (4^3)^2 - (2^6)^2 = (4^3)^2 - (4^3)^2 = 0 \therefore$ A

4. $\sqrt{2 + 3\sqrt{4}} = \sqrt{2+6} = \sqrt{8} = 2\sqrt{2} \therefore$ A

5. $a = 3$, $a + 2b = 3 + 2b = 7 \therefore b = 2$, $2a + 3b + 4l = 6 + 6 + 4l = 16 \therefore l = 1 \therefore$ C

6. $\frac{q-1}{2} = 1 \therefore q = 3$ and $\frac{p+6}{2} = 4 \therefore p = 2 \therefore pq = 2 \times 3 = 6 \therefore$ C

7. Each of A, B, D, E have parts decreasing whereas C is always increasing $\therefore$ C

8. $\frac{0.20}{0.25} + \frac{0.25}{0.20} = \frac{20 \div 5}{25 \div 5} + \frac{25 \div 5}{20 \div 5} = \frac{4}{5} + \frac{5}{4} = \frac{16+25}{20} = \frac{41}{20} = 2\frac{1}{20} = 2.05 \therefore$ A

9. $\triangle ABD \cong \triangle ADE \cong \triangle AEC$ (SSS) and so
$\angle ABD + \angle ADB + \angle ADE + \angle AED + \angle AEC + \angle ACE + \angle BAC$
$= 6\angle ADE + 60^\circ = 3 \times 180^\circ = 540^\circ$ (angle sum of pentagon)

$\angle ADE = \frac{540^\circ - 60^\circ}{6} = 80^\circ \therefore$ E

10. 10^x is an increasing function and $-1000 < -30 < \frac{1}{1000} < \frac{300}{1000} = \frac{3}{10} < 1000$

so $10^{-1000} < 10^{-30} = 1000^{-10} < 10^{\frac{1}{1000}} = \sqrt[1000]{10} < 10^{\frac{3}{10}} = \sqrt[10]{1000} < 10^{1000} = (-10)^{1000} \therefore$ B

11. $12345 + 23451 + 34512 + 45123 + 51234 + 54321 + 43215 + 32154 + 21543 + 15432 = 333330$
maximum digit is $3 \therefore$ B

12.
$$\begin{aligned}
1 + \frac{1}{1-\frac{1}{n}} &= 2.025 \\
\frac{1}{1-\frac{1}{n}} &= 1.025 \\
&= \frac{41}{40} \\
1 - \frac{1}{n} &= \frac{40}{41} \\
\frac{1}{n} &= 1 - \frac{40}{41} \\
&= \frac{1}{41} \\
n &= 41 \therefore \text{ E}
\end{aligned}$$

13. $\angle MXY + \angle NYX = 180^\circ$ (cointerior angles)

$\angle QXY + \angle QYX = 90^\circ$ (complementary angles)

$\therefore \angle MXQ + \angle QYN = \angle MXY + \angle NYX + \angle QXY + \angle QYX = 180^\circ + 90^\circ = 270^\circ \therefore$ D

14. $\frac{\pi r^2}{\pi(1.5r)^2 - \pi r^2} = \frac{1}{1.5^2 - 1} = \frac{1}{1.25} = \frac{4}{5}$ so ratio is $4 : 5 \therefore$ D

15.

Method 1. Table

With $x \in \{0,1,2\}, y \in \{0,1,2,3,4,5,6\}, z \in \{0,1,2,3,4,5,6,7,8,9,10\}$,
$p = 500x + 200y + 50z, q = 36x + 16y + 10z$ and $q \leq 100$ consider the table

x	y	z	p	q	x	y	z	p	q	x	y	z	p	q
0	0	10	500	100	1	1	3	850	82	1	0	1	550	46
0	0	9	450	90	1	0	3	650	66	0	5	1	1050	90
0	1	8	600	96	0	4	3	950	94	0	4	1	850	74
0	0	8	400	80	0	3	3	750	78	0	3	1	650	58
0	1	7	550	86	0	2	3	550	62	0	2	1	450	42
0	0	7	350	70	0	1	3	350	46	0	1	1	250	26
1	0	6	800	96	0	0	3	150	30	0	0	1	50	10
0	2	6	700	92	2	0	2	1100	92	2	1	0	1200	88
0	1	6	500	76	1	2	2	1000	88	2	0	0	1000	72
0	0	6	300	60	1	1	2	800	72	1	4	0	1300	100
1	0	5	750	86	1	0	2	600	56	1	3	0	1100	84
0	3	5	850	98	0	5	2	1100	100	1	2	0	900	68
0	2	5	650	82	0	4	2	900	84	1	1	0	700	52
0	1	5	450	66	0	3	2	700	68	1	0	0	500	36
0	0	5	250	50	0	2	2	500	52	0	6	0	1200	96
1	1	4	900	92	0	1	2	300	36	0	5	0	1000	80
1	0	4	700	76	0	0	2	100	20	0	4	0	800	64
0	3	4	800	88	2	1	1	1250	98	0	3	0	600	48
0	2	4	600	72	2	0	1	1050	82	0	2	0	400	32
0	1	4	400	56	1	3	1	1150	94	0	1	0	200	16
0	0	4	200	40	1	2	1	950	78	0	0	0	0	0
1	2	3	1050	98	1	1	1	750	62					

We see from this maximum p is 1300 when $x = 1, y = 4, z = 0 \therefore$ D

Method 2. Unbounded Knapsack Problem

We may more efficiently find the maximum of $500x+200y+50z$ given that $36x+16y+10z \leq 100$ and x, y, z are non-negative integers with an Unbounded Knapsack Problem via a python script as follows

```python
def solve_knapsack():
    W = 100
    items = [(500, 36, 'x'), (200, 16, 'y'), (50, 10, 'z')]
    dp = [(0, 0, 0, 0)] * (W + 1)
    for w in range(W + 1):
        for val, weight, label in items:
            if w >= weight:
                prev_val, px, py, pz = dp[w - weight]
                new_val = prev_val + val
                if new_val > dp[w][0]:
                    if label == 'x':
                        dp[w] = (new_val, px + 1, py, pz)
                    elif label == 'y':
                        dp[w] = (new_val, px, py + 1, pz)
                    else:
```

```
                        dp[w] = (new_val, px, py, pz + 1)
    max_val = 0
    best_config = (0, 0, 0, 0)
    for i in range(W + 1):
        if dp[i][0] > max_val:
            max_val = dp[i][0]
            best_config = (max_val, dp[i][1], dp[i][2], dp[i][3])
    return best_config
print(solve_knapsack())
```

This returns $(1300, 1, 4, 0)$ which means we have 1 pack of 500 and 4 packs of 200 to get maximum 1300 $\therefore$ D

16. Where x is side length of regular hexagon, y is width of the strip then $x + \frac{2y}{\sqrt{3}} = 20$ and $x - \frac{4y}{\sqrt{3}} = 10$ so $\frac{6y}{\sqrt{3}} = 10$ and $y = \frac{10\sqrt{3}}{6}$

Now $6x = 6(20 - \frac{20\sqrt{3}}{6\sqrt{3}}) = 100 \therefore$ D

17. As $p+t+x$ is unique it matters not if each variable is unique. Hence one example suffices. Such an example is $(p,q,r,s,t,u,v,w,x) = (5,7,6,8,9,3,4,2,1)$ and $p+t+x = 15 \therefore$ B.

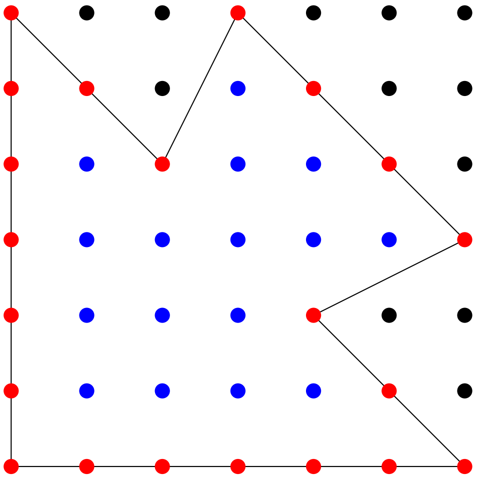
18.

Method 1. Shoelace formula

If we place the square w.l.o.g. with side length 1 in a Cartesian plane such that $D = (0,0)$, $C = (1,0), B = (1,1), A = (0,1)$ then $M = (\frac{1}{2}, 1), N = (1, \frac{1}{2})$. If P and Q are respectively the intersections of DM (on line $y = 2x$) and DN (on line $y = \frac{1}{2}x$) with AC (on line $y = 1 - x$) then for coordinates of $P, 2x = 1 - x$ so $x = \frac{1}{3}$ and $y = \frac{2}{3}$ and likewise for $Q . \frac{1}{2}x = 1 - x$ so $x = \frac{2}{3}$ and $y = \frac{1}{3}$. Hence $P = (\frac{1}{3}, \frac{2}{3})$ and $Q = (\frac{2}{3}, \frac{1}{3})$. By the shoelace formula applied to polygon $DCQNMPA$,

$$\begin{aligned}\text{Shaded area} &= \tfrac{1}{2}\left(\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} + \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ 1 & \frac{1}{2} \end{vmatrix} + \begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} + \begin{vmatrix} \frac{1}{2} & 1 \\ \frac{1}{3} & \frac{2}{3} \end{vmatrix} + \begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}\right) \\ &= \tfrac{1}{2}(0 + \tfrac{1}{3} + 0 + \tfrac{3}{4} + 0 + \tfrac{1}{3} + 0) \\ &= \tfrac{17}{24} \therefore \text{ C}\end{aligned}$$

Method 2. Pick's Theorem.



$\frac{1}{36}(16 + \frac{21}{2} - 1) = \frac{17}{24} \therefore$ C.

Method 3. Cross Products.

Using coordinates from Method 1 we have

$$\begin{aligned}
&\tfrac{1}{2}(|\overrightarrow{DC}\times\overrightarrow{DA}|+|\overrightarrow{DN}\times\overrightarrow{DM}|-|\overrightarrow{DQ}\times\overrightarrow{DP}|)\\
&=\tfrac{1}{2}(|\vec{i}\times\vec{j}|+|(\vec{i}+\tfrac{1}{2}\vec{j})\times(\tfrac{1}{2}\vec{i}+\vec{j})|-|(\tfrac{2}{3}\vec{i}+\tfrac{1}{3}\vec{j})\times(\tfrac{1}{3}\vec{i}+\tfrac{2}{3}\vec{j})|)\\
&=\tfrac{1}{2}(|\vec{k}|+|\tfrac{3}{4}\vec{k}|-|\tfrac{1}{3}\vec{k}|)\\
&=\tfrac{1}{2}(1+\tfrac{3}{4}-\tfrac{1}{3})\\
&=\tfrac{17}{24}\therefore \text{C.}
\end{aligned}$$

19. $a=\tan^{-1}\frac{1}{3}$ so $a+b=\tan^{-1}\frac{1}{3}+b=\tan^{-1}\frac{2}{3}$ and $b=\tan^{-1}\frac{2}{3}-\tan^{-1}\frac{1}{3}$

Now $\tan b=\frac{\frac{2}{3}-\frac{1}{3}}{1+\frac{2}{3}\times\frac{1}{3}}=\frac{3}{11}$ and so $b=\tan^{-1}\frac{3}{11}$

$a+b+c=\tan^{-1}\frac{2}{3}+c=\tan^{-1}\frac{3}{3}=\tan^{-1}1$ so $c=\tan^{-1}1-\tan^{-1}\frac{2}{3}$

Now $\tan c=\frac{1-\frac{2}{3}}{1+1\times\frac{2}{3}}=\frac{1}{5}$ and so $c=\tan^{-1}\frac{1}{5}$

$a+b+c+d=\tan^{-1}1+d=\tan^{-1}\frac{4}{2}=\tan^{-1}2$ so $d=\tan^{-1}2-\tan^{-1}1$

Now $\tan d=\frac{2-1}{1+2\times1}=\frac{1}{3}$ and so $d=\tan^{-1}\frac{1}{3}=a$

$a+b+c+d+e=\tan^{-1}2+e=\tan^{-1}\frac{5}{1}=\tan^{-1}5$ so $e=\tan^{-1}5-\tan^{-1}2$

Now $\tan e=\frac{5-2}{1+5\times2}=\frac{3}{11}$ and so $e=\tan^{-1}\frac{3}{11}=b$

$f=\tan^{-1}\frac{1}{5}=c$

So $a=d, b=e, c=f\ \therefore$ A

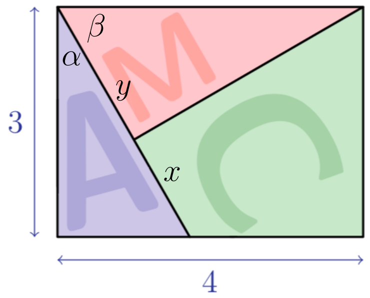
20. Surface area of original cone is $2\pi(2+4)=12\pi$ cm^2 and height $\sqrt{4^2-2^2}=2\sqrt{3}$ cm. If the cut is made h cm below the apex the small cone has base radius r then $\frac{h}{r}=\frac{2\sqrt{3}}{2}=\sqrt{3}$ so $r=\frac{h}{\sqrt{3}}$. If slant height of small cone is l then $\frac{4}{2\sqrt{3}}=\frac{l}{h}$ so $l=\frac{2h}{\sqrt{3}}$ and slant height of frustrum of cone is $4-\frac{2h}{\sqrt{3}}\ \therefore\ \frac{\pi h}{\sqrt{3}}(\frac{h}{\sqrt{3}}+\frac{2h}{\sqrt{3}})=\pi((2+\frac{h}{\sqrt{3}})(4-\frac{2h}{\sqrt{3}})+2^2+(\frac{h}{\sqrt{3}})^2)$
$h^2=12-\frac{h^2}{3}\ \therefore\ \frac{4h^2}{3}=12\ \therefore\ h^2=9$ and $h=3\ \therefore$ E

21. $PS=x\Rightarrow PT=\sqrt{169-x^2}\ \therefore\ \frac{1}{2}x\sqrt{169-x^2}=\frac{1}{3}x^2\ \therefore\ (3\sqrt{169-x^2})^2=(2x)^2$

$9(169-x^2)=4x^2\ \therefore\ 13x^2=9\times169\ \therefore\ x^2=9\times13=117\ \therefore$ B

22. $2^9-10=502\ \therefore$ E.

23.



$(x+y)^2 = 3 \times 4 = 12 \therefore \alpha = \tan^{-1}\frac{\sqrt{(x+y)^2-3^2}}{3} = \tan^{-1}\frac{\sqrt{12-9}}{3} = \tan^{-1}\frac{\sqrt{3}}{3} = \frac{\pi}{6}$

$\beta = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$ so $y = 4\cos\frac{\pi}{3} = 2$

$x + y = \sqrt{12} = 2\sqrt{3}$ so $x = 2\sqrt{3} - y = 2\sqrt{3} - 2 = 2(\sqrt{3}-1) \therefore$ E.

24. $$\begin{aligned} \tfrac{60x}{105} + \tfrac{60-60x}{35} &= 1 \\ (\tfrac{12}{7} - \tfrac{4}{7})x &= \tfrac{12}{7} - 1 \\ 8x &= 5 \\ x &= \tfrac{5}{8} \therefore \text{ C} \end{aligned}$$

25. $\frac{2x^2(1+\sqrt{2})}{4x^2(1+\sqrt{2})+2x^2} = \frac{1+\sqrt{2}}{3+2\sqrt{2}} \cdot \frac{3-2\sqrt{2}}{3-2\sqrt{2}} = \frac{3+3\sqrt{2}-2\sqrt{2}-4}{9-8} = \sqrt{2} - 1 \therefore$ A

26. $P(2) = 32a + 16b + 8c + 4d + 2e + f = 11$.
If $a = -1, \max(P(2) : b, c, d, e, f \in \{-1, 1\}) = -32 + 16 + 8 + 4 + 2 + 1 = -1 < 11$ so $a \neq -1 \therefore a = 1$ and $32 + 16b + 8c + 4d + 2e + f = 11$ so $16b + 8c + 4d + 2e + f = -21$

If $b = 1, \min(16 + 8c + 4d + 2e + f : c, d, e, f \in \{-1, 1\}) = 16 - 8 - 4 - 2 - 1 = 1 > -21$ so $b = -1$ and $-16 + 8c + 4d + 2e + f = -21$ and $8c + 4d + 2e + f = -5$

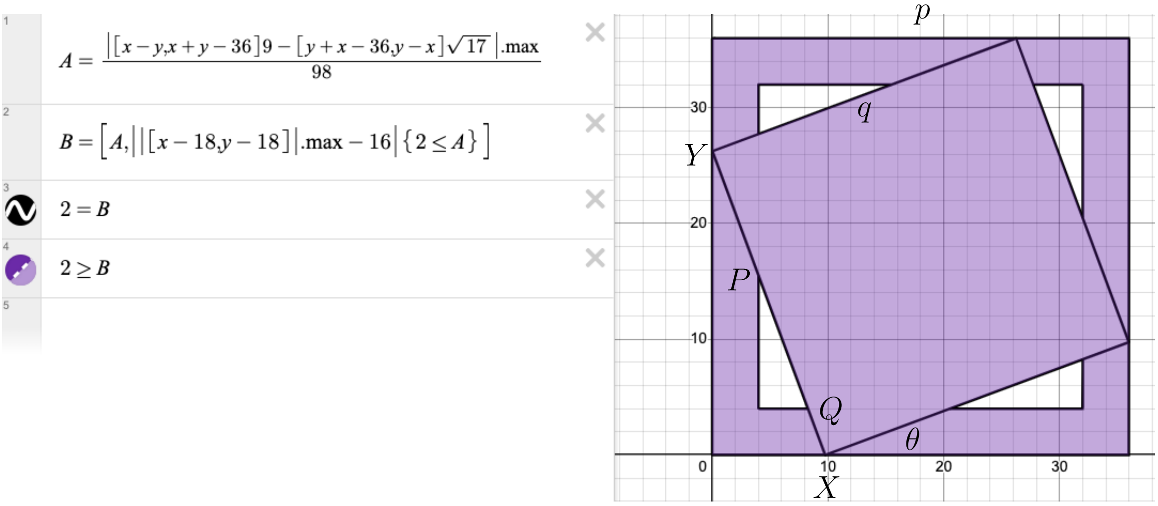
If $c = 1, \min(8 + 4d + 2e + f : d, e, f \in \{-1, 1\}) = 8 - 4 - 2 - 1 = 1 > -5$ so $c = -1$ and $-8 + 4d + 2e + f = -5$ so $4d + 2e + f = 3$

If $d = -1, \max(-4 + 2e + f : e, f \in \{-1, 1\}) = -4 + 2 + 1 = -1 < 3 \therefore d = 1$ and $4 + 2e + f = 3$ and $2e + f = -1$

If $e = 1, \min(2 + f : f \in \{-1, 1\}) = 2 - 1 = 1 > -1$ so $e = -1$ and $-2 + f = -1$ and $f = 1$

Now $P(x) = x^5 - x^4 - x^3 + x^2 - x + 1$ so $P(4) = 4^5 - 4^4 - 4^3 + 4^2 - 4 + 1 = 717$

27. One may recreate the diagram in desmos as follows:



$$A = \frac{\left|[x-y, x+y-36]9 - [y+x-36, y-x]\sqrt{17}\right|.\max}{98}$$

$$B = \left[A, \left|\left|[x-18, y-18]\right|.\max - 16\right| \{2 \le A\}\right]$$

$$2 = B$$

$$2 \ge B$$

Let S_1 be the large square with side length p, S_2 be the small square with side length q, S_3 be S_2 rotated as in the diagram in the question anticlockwise by θ and have x- and y-intercepts $X(b,0), Y(0,a)$. Then using Chebyshev distance from centre and rotation matrix we have where $\theta = \tan^{-1}\frac{b}{a}$,

$S_1 : \max(|2x-p|, |2y-p|) = p$,
$S_2 : \max(|2x-p|, |2y-p|) = q$ and
$S_3 : \max\left(\left|\begin{vmatrix} 2x-p & 2y-p \\ -\sin\theta & \cos\theta \end{vmatrix}\right|, \left|\begin{vmatrix} 2x-p & 2y-p \\ \cos\theta & \sin\theta \end{vmatrix}\right|\right) = q$

Also $a+b=p, a^2+b^2=q^2, a>b \Rightarrow a = \frac{p+\sqrt{2q^2-p^2}}{2}, b = \frac{p-\sqrt{2q^2-p^2}}{2}$

XY intersecting S_2 at P and Q where $x = \frac{p-q}{2}, y = \frac{p-q}{2}$ respectively by substituting into S_3 we find $P = \left(\frac{p-q}{2}, \frac{p^2-q^2+pq+q\sqrt{2q^2-p^2}}{2(p+q)}\right), Q = \left(\frac{p^2-q^2+pq-q\sqrt{2q^2-p^2}}{2(p+q)}, \frac{p-q}{2}\right)$

Now white area $= 2\left(\frac{p^2-q^2+pq+q\sqrt{2q^2-p^2}}{2(p+q)} - \frac{p-q}{2}\right)\left(\frac{p^2-q^2+pq-q\sqrt{2q^2-p^2}}{2(p+q)} - \frac{p-q}{2}\right) = \frac{q^2(p-q)}{p+q}$

With $p=36, q=28$, white area $= \frac{28^2(36-28)}{36+28} = 98$

28. $\frac{1}{3}\cdot\frac{3\sqrt{3}}{2}\cdot\left(\left(\frac{10}{2}\right)^2+\left(\frac{10}{2}\right)^2\right)\cdot\frac{1}{2}\cdot\sqrt{10^2+10^2+10^2} = 375$

29. For n diagonals we have $3\sum_{i=0}^{n}\sum_{j=0}^{n-i} ij = \frac{(n+2)!}{8(n-2)!}$ parallelograms so with $n=9$ number of parallelograms is $\frac{11!}{8!} = 990$

30. Considering $p(x)-x$ as $p(1)-1=0$ and $p(10)-10=0$ then $x-1, x-10 | p(x)-x$ so $p(x) = (x-1)(x-10)q(x)+x$ for a quadratic $q(x)$.

$q(x) = k(x^2-sx+r)$ for integers k, s, r and $p(n)-n \ge 0$ for all integers $n \Rightarrow$ for large $x, (x-1)(x-10) > 0$ so for large x, $q(x) \ge 0$ and so $k > 0$

$p(-3) = k(-4)(-13)(9+3s+r)-3 = 52k(9+3s+r)-3 = 2025 \therefore k(9+3s+r) = 39$

So $k \in \{1, 3, 13, 39\}$ and $r = \frac{39}{k} - 3s - 9$ for integers s.

We also have $q(x) < 0$ for $2 \le x \le 9$ and $q(x) > 0$ for $x < 0$ or $x > 10$ and with these conditions $k=1, s=10, r=0$ and so $q(x) = x(x-10)$

Now $p(x) = x(x-1)(x-10)^2 + x$ so $p(3) = 3(2)(-7)^2 + 3 = 297$