troids and axoids (*Liouville's Journal*, XVI, 9–129; 286–336). The application of these methods to the study of Mechanisms was first made systematically by Willis (1841); his processes were improved by Rankine (*Machinery and Millwork*, 1871); and the whole subject was reformed and freshly stated by Reuleaux (*Kinematik*, 1874), and has since been still further developed by Grashof and others. Peculiar interest attaches to the "tram-motion" from its furnishing to Leonardo da Vinci his famous discovery of the elliptic chuck for turning ovals on the lathe (Chasles, *Aperçu* 531).—*W. M. T.*]

A COLLECTION OF FORMULÆ FOR THE AREA OF A PLANE TRIANGLE.*

By Mr. Marcus Baker, Washington, D. C.

In April, 1883, Mr. James Main, formerly of the U. S. Coast and Geodetic Survey, published in the *Mathematical Magazine* a collection of forty-six (46) expressions for the area of a plane triangle, prefacing it with the remark that this collection "may be regarded as a matter of curiosity," and that about one-half of the formulæ are well known.

In the following August M. Ed. Lucas reprinted this collection in *Mathesis* in a classified form, separating the formulæ into five groups and adding one formula not contained in Mr. Main's list. The collection has also been reprinted in the third number of the *Tidsskrift for Mathematik*, 1883. Some two or three additional formulæ have since been printed in various mathematical publications.

The terms in which the area is expressed in Mr. Main's collection are angles, sides, perpendiculars, and radii of inscribed, escribed, and circumscribed circles. No formulæ are given involving medians or bisectors. In numbering Mr. Main has not counted those formulæ as distinct which arise from merely permuting the letters, nor has he in *every* case given all the forms possible to be obtained by permuting the letters, though he has generally done so. As numbered, then, he counts forty-six formulæ, but if every form be counted as a distinct one the total number is ninety-four.

M. Lucas, by making all possible permutations and adding one new form, makes the number 139, to which some two or three have been added since.

As the matter has proved of interest, the following collection has been made, which is a still further extension; the additional formulæ being chiefly due to introducing the medians and bisectors, not used in the former collections. In this collection Mr. Main's mode of numbering has been followed and formulæ

*Read before the Mathematical section of the Philosophical Society of Washington, January 7, 1885.

derived by permutation are not enumerated as distinct formulæ. Moreover, formulæ expressed in the same terms, but in different form, are also considered as but one. For example, in the former lists we find

$$\Delta = 2R^2 \sin A \sin B \sin C,$$

$$\Delta = \tfrac{1}{2}R^2 (\sin 2A + \sin 2B + \sin 2C),$$

and $$\Delta = \tfrac{2}{3}R^2 [\sin^3 A \cos(B - C) + \sin^3 B \cos (C - A) + \sin^3 C \cos (A - B)]$$

given as three distinct formulæ whereas they are here counted as *one*, and the principle involved herein is employed throughout.

The total number of formulæ for the area of a plane triangle in this collection is ninety-three, not counting those arising from permutation. If these be counted as distinct the total number is two hundred and sixty-nine.

Owing to pressure of other duties and consequent lack of time some of the groups in this collection are not so fully worked up as had been planned.

It may be noted that a number of curious and interesting theorems may be obtained by equating different expressions for the area and reducing. In this collection we have classified, for convenience, all expressions for the area into five groups and each group into two parts.

Group I contains formulæ which M. Lucas has called *unique*, i. e. formulæ which do not admit of other similar formulæ by merely permuting the letters. In such formulæ *all* the sides, *all* the perpendiculars, *all* the medians, etc. must enter if one enters.

Group II contains formulæ which admit of *two* similar expressions by permutation giving three of a kind.

Group III contains formulæ which admit of *three* similar expressions by permutation giving four of a kind.

Group IV contains formulæ which admit of *five* similar expressions by permutation giving six of a kind.

Group V contains formulæ which admit of *eleven* similar expressions by permutation giving twelve of a kind.

Each group is divided into *two* parts, the *first containing* and the *second not containing* trigonometrical functions. In addition to the foregoing a group of miscellaneous expressions, not falling within the classification used, has been added, and called the Miscellaneous Group.

The notation used is as follows:

Δ = the area of the triangle;
A, B, C = the angles;
a, b, c = the sides opposite A, B, and C respectively
s = semi perimeter = $\frac{1}{2}(a + b + c)$;

R, r, r_a, r_b, r_c = the radii of circumscribed, inscribed, escribed circles respectively;
h_a, h_b, h_c = the perpendiculars from A, B, and C respectively;
m_a, m_b, m_c = the medians from A, B, and C respectively;
${}_i\beta_a$, ${}_i\beta_b$, ${}_i\beta_c$ = the bisectors (internal) of the angles A, B, and C respectively; and
$\sigma = \frac{1}{2}(m_a + m_b + m_c)$.

GROUP I. PART I.

$\Delta =$

1. $\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4}$

2. $\frac{4}{3}\sqrt{\sigma(\sigma - m_a)(\sigma - m_b)(\sigma - m_c)}$
$= \frac{1}{3}\sqrt{2m_a^2m_b^2 + 2m_b^2m_c^2 + 2m_c^2m_a^2 - m_a^4 - m_b^4 - m_c^4}$

3. $1 \div$
$$\sqrt{\left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right)\left(-\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right)\left(\frac{1}{h_a} - \frac{1}{h_b} + \frac{1}{h_c}\right)\left(\frac{1}{h_a} + \frac{1}{h_b} - \frac{1}{h_c}\right)}$$
$= h_a^2h_b^2h_c^2 \div$
$$\sqrt{(h_ah_b + h_bh_c + h_ch_a)(-h_ah_b + h_bh_c + h_ch_a)(h_ah_b - h_bh_c + h_ch_a)(h_ah_b + h_bh_c - h_ch_a)}$$

4. $\sqrt{rr_ar_br_c}$

5. $\frac{abc}{4R}$

6. $\frac{2}{27R}\sqrt{(2m_a^2 + 2m_b^2 - m_c^2)(2m_b^2 + 2m_c^2 - m_a^2)(2m_c^2 + 2m_a^2 - m_b^2)}$

7. $\sqrt{\frac{1}{2}Rh_ah_bh_c}$

8. $\frac{r}{2}\sqrt{\frac{(r_a + r_b)(r_b + r_c)(r_c + r_a)}{R}}$

9. $\frac{1}{2}\sqrt[3]{abch_ah_bh_c}$

10. $$\frac{abc}{r_a + r_b + r_c - r} = \frac{r^2(r_a + r_b)(r_b + r_c)(r_c + r_a)}{abc} = \left\{\frac{a^2 + b^2 + c^2}{\frac{1}{r^2} + \frac{1}{r_a^2} + \frac{1}{r_b^2} + \frac{1}{r_c^2}}\right\}^{\frac{1}{2}}$$

11. $$\sqrt{\tfrac{1}{3}(m_a^2 + m_b^2 + m_c^2)} \div \left(\frac{1}{h_a^2} + \frac{1}{h_b^2} + \frac{1}{h_c^2}\right)$$
$$= \frac{1}{3h_ah_bh_c}\sqrt{3(m_a^2 + m_b^2 + m_c^2)(h_a^2h_b^2 + h_b^2h_c^2 + h_c^2h_a^2)}$$

12. $2R^2\frac{h_ah_bh_c}{abc} = \frac{1}{2}\frac{R}{s}(h_ah_b + h_bh_c + h_ch_a)$

GROUP I. PART I.—*Continued.*

$\Delta =$

13. $\left(\frac{R^2}{12}(m_a^2 + m_b^2 + m_c^2)(h_a^2h_b^2 + h_b^2h_c^2 + h_c^2h_a^2)\right)^{\frac{1}{6}}$

14. $\frac{\beta_a\beta_b\beta_c}{s} \cdot \frac{a+b}{2c} \cdot \frac{b+c}{2a} \cdot \frac{c+a}{2b} = \frac{1}{4}\beta_a\beta_b\beta_c\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{1}{a+b+c}\right)$

15. $\frac{3}{8}\beta_a\beta_b\beta_c\left[\frac{1}{\sqrt{2m_a^2 + 2m_b^2 - m_c^2}} + \frac{1}{\sqrt{2m_b^2 + 2m_c^2 - m_a^2}} + \frac{1}{\sqrt{2m_c^2 + 2m_a^2 - m_b^2}}\right.$

$\left. - \frac{1}{\sqrt{2m_a^2 + 2m_b^2 - m_c^2} + \sqrt{2m_b^2 + 2m_c^2 - m_a^2} + \sqrt{2m_c^2 + 2m_a^2 - m_b^2}}\right]$

16. $\frac{1}{2}\sqrt{\frac{1}{2}\beta_a\beta_b\beta_c\left\{h_a + h_b + h_c - \frac{1}{\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}}\right\}}$

17. $\frac{1}{2}\sqrt{\beta_a\beta_b\beta_c\left\{\frac{1}{\frac{1}{r_a} + \frac{1}{r_b}} + \frac{1}{\frac{1}{r_b} + \frac{1}{r_c}} + \frac{1}{\frac{1}{r_c} + \frac{1}{r_a}} - \frac{1}{r}\right\}}$

18. $\sqrt{\frac{1}{8}r\beta_a\beta_b\beta_c\left[(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 1\right]}$

19. $\frac{1}{4}\sqrt{\frac{\beta_a\beta_b\beta_c}{R} \cdot \frac{(a+b)(b+c)(c+a)}{a+b+c}}$

20. $\frac{Rr}{\beta_a\beta_b\beta_c}\left(\frac{1}{a} + \frac{1}{b}\right)\left(\frac{1}{b} + \frac{1}{c}\right)\left(\frac{1}{c} + \frac{1}{a}\right)$

21. $\frac{h_ah_bh_c}{8abc}(r_a + r_b + r_c - r)^2.$

GROUP I. PART II.

22. $2R^2 \sin A \sin B \sin C$
$= \frac{1}{2}R^2(\sin 2A + \sin 2B + \sin 2C)$
$= \frac{2}{3}R^2[\sin^3 A \cos(B - C) + \sin^3 B \cos(C - A) + \sin^3 C \cos(A - B)]$

23. $\frac{1}{2}R(a \cos A + b \cos B + c \cos C)$
$= R[a \cos C \cos A + b \cos A \cos B + c \cos B \cos C]$

24. $\frac{1}{4}(a^2 \cot A + b^2 \cot B + c^2 \cot C) = \frac{1}{4} \cdot \frac{a^2 + b^2 + c^2}{\cot A + \cot B + \cot C}$

GROUP I. PART II.—*Continued.*

$\Delta =$

25.* $\frac{1}{3}\dfrac{m_a^2 + m_b^2 + m_c^2}{\cot A + \cot B + \cot C}$

26. $\sqrt{2R_i\mathfrak{z}_{a_i}\mathfrak{z}_{b_i}\mathfrak{z}_c\left[1 + \cos(A - B) + \cos(B - C) + \cos(C - A)\right]}$

$= 2\sqrt{R_i\mathfrak{z}_{a_i}\mathfrak{z}_{b_i}\mathfrak{z}_c \sin(A + \frac{1}{2}B)\sin(B + \frac{1}{2}C)\sin(C + \frac{1}{2}A)}$

27. $R\sqrt[3]{h_a h_b h_c \sin A \sin B \sin C}$

28. $\frac{1}{3}R \sin A \sin B \sin C \left(\dfrac{ab}{h_c} + \dfrac{bc}{h_a} + \dfrac{ca}{h_b}\right)$

29. $\dfrac{h_a + h_b + h_c}{2\left(\dfrac{\cos\frac{1}{2}A}{\mathfrak{z}_a} + \dfrac{\cos\frac{1}{2}B}{\mathfrak{z}_b} + \dfrac{\cos\frac{1}{2}C}{\mathfrak{z}_c}\right)}$

30. $\dfrac{\mathfrak{z}_a \sin(C + \frac{1}{2}A) + \mathfrak{z}_b \sin(A + \frac{1}{2}B) + \mathfrak{z}_c \sin(B + \frac{1}{2}C)}{2\left(\dfrac{\cos\frac{1}{2}A}{\mathfrak{z}_a} + \dfrac{\cos\frac{1}{2}B}{\mathfrak{z}_b} + \dfrac{\cos\frac{1}{2}C}{\mathfrak{z}_c}\right)}$

31. $\dfrac{\mathfrak{z}_{a_i}\mathfrak{z}_{b_i}\mathfrak{z}_c\sqrt{(m_a^2 - h_a^2)(m_b^2 - h_b^2)(m_c^2 - h_c^2)}}{s(a - b)(b - c)(c - a)}.$

[TO BE CONTINUED].

DEMONSTRATION OF DESCARTES'S THEOREM AND EULER'S THEOREM.

By PROF. G. B. HALSTED, Austin, Texas.

DESCARTES'S THEOREM.

Cutting by diagonals the faces not triangles into triangles, the whole surface of any polyhedron contains a number of triangular faces four less than double the number of summits.

Proof.

For, joining all the summits by a single closed broken line, this cuts the surface into two skew polygons, each of which contains $S - 2$ triangles, where S is the number of summits.

* *Tidsskrift for Mathematik*, 8° Copenhagen, 1883, fifth series, first year, No. 4, p. 136.

we obtain as their internal product

$$\begin{aligned} a \mid b &= (\alpha_1 e_1 + \alpha_2 e_2)(\beta_1 e_2 - \beta_2 e_1) \\ &= (\alpha_1\beta_1 + \alpha_2\beta_2)\, e_1 e_2 \\ &= \alpha_1\beta_1 + \alpha_2\beta_2. \end{aligned}$$

Similarly we find for the internal square

$$a \mid a = \alpha_1^2 + \alpha_2^2.$$

[TO BE CONTINUED.]

A COLLECTION OF FORMULÆ FOR THE AREA OF A PLANE TRIANGLE.

By MR. MARCUS BAKER, Washington, D. C.

[CONTINUED FROM VOL. I, PAGE 138.]

GROUP II. PART I.

$\Delta =$

32. $\frac{1}{4}\sqrt{8m_c^2(a^2+b^2-2m_c^2)-(a^2-b^2)^2}$

33. $\frac{1}{2}h_c(\sqrt{a^2-h_c^2}+\sqrt{b^2-h_c^2})$

34. $\frac{1}{2}\beta_c ab\left(\frac{1}{a}+\frac{1}{b}\right)\sqrt{1-\frac{1}{4}\beta_c^2\left(\frac{1}{a}+\frac{1}{b}\right)^2}$

35. $\frac{1}{3}\sqrt{4m_b^2m_c^2-[\frac{9}{4}a^2-(m_b^2+m_c^2)]^2}=\frac{2}{3}\sqrt{m_b^2m_c^2-k_a^4}$

where $2k_a^2=\frac{9}{4}a^2-(m_b^2+m_c^2)$, etc.

36. $\dfrac{h_b^3\sqrt{a^2-h_b^2}-h_c^3\sqrt{a^2-h_c^2}}{2(h_b^2-h_c^2)}$

37. $\frac{1}{6}h_a(\sqrt{4m_b^2-h_a^2}+\sqrt{4m_c^2-h_a^2})$

38. $\frac{1}{2}ah_a.$

39. $\frac{1}{2}\sqrt{abh_ah_b}=\frac{1}{2}\dfrac{a\pm b}{\frac{1}{h_a}\pm\frac{1}{h_b}}$

GROUP II. PART 1.—*Continued.*

$\Delta =$

40.* $h_a \sqrt{\left(h_a^2 + \sqrt{(m_a^2 - h_a^2)(\beta_a^2 - h_a^2)}\right)\left(\sqrt{\frac{m_a^2 - h_a^2}{\beta_a^2 - h_a^2}} - 1\right)}$

41. $\frac{1}{2}h_a(\sqrt{\frac{1}{4}a^2 + m_a^2 - h_a^2 + a\sqrt{m_a^2 - h_a^2}} + \sqrt{\frac{1}{4}a^2 + m_a^2 - h^2 - a\sqrt{m_a^2 - h_a^2}})$

$= \frac{1}{2}h_a(\sqrt{\frac{1}{4}a^2 + k_a^2 + ak} + \sqrt{\frac{1}{4}a^2 + k_a^2 - ak})$, where $k_a = m_a^2 - h_a^2$, etc.

42. $\frac{Rh_bh_c}{a}$

43. $\frac{ab}{4R}\sqrt{2(a^2 + b^2 - 2m_c^2)}$

44. $h_a(\sqrt{b^2 - h_a^2} \pm \sqrt{m_a^2 - h_a^2})$

45.† $\frac{ab}{8R^2}(a\sqrt{4R^2 - b^2} + b\sqrt{4R^2 - a^2})$

46. $2R^2[(f - 2x^2)\sqrt{(g - 2x^2)(1 - g + 2x^2)}$

$+ (g - 2x^2)\sqrt{(f - 2x^2)(1 - f + 2x^2)}]$,

where $f = \frac{m_a^2 + 2m_b^2}{3R^2}$, $g = \frac{2m_a^2 + m_b^2}{3R^2}$, and $x\ (= \sin C)$ is to be found from the equation

$x = \sqrt{(f - 2x^2)(1 - g + 2x^2)} + \sqrt{(g - 2x^2)(1 - f + 2x^2)}$.

47. $\frac{sh_ar_a}{2(h_a + r_a)}$

*If μ and ϕ are auxiliaries determined from the equations $\cos\mu = \frac{h_a}{m_a}$ and $\cos\phi = \frac{h_a}{\beta_a}$, then

$$\Delta = \frac{h_a^2}{\cos\mu}\sqrt{\frac{\sin 2(\mu - \phi)}{\sin 2\phi}}.$$

[For this elegant expression I am indebted to my friend Mr. Charles H. Kummell, of the U. S. Coast and Geodetic Survey.]

†$\sqrt{R^2h_a^2 - \Delta^2} + \sqrt{R^2h_b^2 - \Delta^2} = \frac{1}{2}\left[\frac{Rh_ah_b}{\Delta}\right]^2$

GROUP II. PART II.

$\Delta =$

48. $\frac{1}{2}ab \sin C$

49. $\frac{h_a h_b}{2 \sin C}$

50. $\frac{1}{2}a^2 \frac{\sin B \sin C}{\sin A} = \frac{\frac{1}{2}a^2}{\cot B + \cot C}$

51. $\frac{1}{2}h_a^2 \frac{\sin A}{\sin B \sin C} = \frac{1}{2}h_a^2 (\cot B + \cot C) = \frac{1}{2}h_a^2 (\sin 2B + \sin 2C)$

52. $\frac{1}{2}(a^2 - b^2) \frac{\sin A \sin B}{\sin (A - B)} = \frac{1}{4}(a^2 \sin 2B + b^2 \sin 2A) = \frac{\frac{1}{2}(a^2 - b^2)}{\cot A - \cot B}.$

53. $\frac{1}{2}(h_b^2 - h_a^2) \frac{\sin A \sin B}{\sin (A - B)} \cdot \frac{1}{\sin^2 C} = \frac{1}{2}h_a h_b \frac{\tan \frac{1}{2}A \tan B \tan \frac{1}{2}C}{\cos A}$

$= h_a h_b \frac{\tan \frac{1}{2}A \tan \frac{1}{2}B \tan \frac{1}{2}C}{\cos A (1 - \tan^2 \frac{1}{2}B)}$

54. $\frac{2m_a^2 \sin A \sin B \sin C}{2 (\sin^2 A + \sin^2 B + \sin^2 C) - 3 \sin^2 A}$

55. $\frac{2 (m_b^2 - m_a^2)}{3 (\cot A - \cot B)}$

56. $\frac{1}{3}\sqrt{4m_b^2 m_c^2 - [9R^2 \sin^2 A - (m_b^2 + m_c^2)]^2}$

57. $\frac{2R^4}{m_a^2} \sin A \sin B \sin C (-\sin^2 A + 2 \sin^2 B + 2 \sin^2 C)$

58. $\frac{4m_a^2 + 3a^2}{8 (\cot A + \cot B + \cot C)}$

59. $\frac{4 (m_a^2 + m_b^2) - 3c^2}{4 (\cot A + \cot B + \cot C)}$

60. $\frac{1}{2}\beta_a^2 \frac{\sin A}{\sin B \sin C} \sin (B + \frac{1}{2}A) \sin (A + \frac{1}{2}B)$

61. $Rh_a \sin A$

62. $\frac{1}{2}a\beta_a \sin (C + \frac{1}{2}A)$

63. $R^2 \sin 2A (1 + \cos C)$

64. $Ra \sin B \sin C = Ra (\cos A + \cos B \cos C)$

65. $Rh_a \tan A \sin B \tan \frac{1}{2}C$

66. $R\beta_a \tan A \sin B \tan \frac{1}{2}C \sin (C + \frac{1}{2}A)$

GROUP II. PART II.—*Continued.*

$\Delta =$

67. $\dfrac{h_a(h_b + h_c)}{4 \sin (C + \frac{1}{2}A) \cos \frac{1}{2}A}$

68. $\dfrac{\beta_a (h_b + h_c)}{4 \cos \frac{1}{2}A}$

69. $\frac{1}{4}\beta_a^2 (b + c) \left(\dfrac{1}{b} + \dfrac{1}{c}\right) \tan \frac{1}{2}A.$

70. $\frac{1}{2}\beta_a\beta_b \dfrac{\sin (B + \frac{1}{2}A) \sin (A + \frac{1}{2}B)}{\sin C}$

71. $\beta_a\beta_b\beta_c \dfrac{-\beta_a \sin \frac{1}{2}A + \beta_b \sin \frac{1}{2}B + \beta_c \sin \frac{1}{2}C}{-\beta_b\beta_c \cos \frac{1}{2}A + \beta_c\beta_a \cos \frac{1}{2}B + \beta_a\beta_b \cos \frac{1}{2}C}$

72. $\frac{1}{2}a \sin B (a \cos B + \sqrt{b^2 - a^2 \sin^2 B}) = \frac{1}{2}b \sin A (b \cos A + \sqrt{a^2 - b^2 \sin^2 A})$

GROUP III. PART I.

73. $sr = (s - a) r_a$

74. $\frac{1}{3}r (\sqrt{2m_a^2 + 2m_b^2 - m_c^2} + \sqrt{2m_b^2 + 2m_c^2 - m_a^2} + \sqrt{2m_c^2 + 2m_a^2 - m_b^2})$
$= \frac{1}{3}r_a (\sqrt{2m_a^2 + 2m_b^2 - m_c^2} - \sqrt{2m_b^2 + 2m_c^2 - m_a^2} + \sqrt{2m_c^2 + 2m_a^2 - m_b^2})$

75. $\sqrt{\dfrac{r r_b r_c}{-\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}}} = \sqrt{\dfrac{r_a r_b r_c}{\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}}}$

76. $\dfrac{R}{2s}(h_a h_b + h_b h_c + h_c h_a) = \dfrac{R}{2(s - a)}(h_a h_b - h_b h_c + h_c h_a)$

77. $\sqrt{\frac{1}{2}Rr (h_a h_b + h_b h_c + h_c h_a)} = \sqrt{\frac{1}{2}Rr_a (h_a h_b - h_b h_c + h_c h_a)}$

78. $\dfrac{s}{\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}} = \dfrac{s - a}{-\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}}$

79. $\sqrt{\dfrac{r}{\left(-\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right)\left(\frac{1}{h_a} - \frac{1}{h_b} + \frac{1}{h_c}\right)\left(\frac{1}{h_a} + \frac{1}{h_b} - \frac{1}{h_c}\right)}}$
$= \sqrt{\dfrac{r_a}{\left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right)\left(\frac{1}{h_a} - \frac{1}{h_b} + \frac{1}{h_c}\right)\left(\frac{1}{h_a} + \frac{1}{h_b} - \frac{1}{h_c}\right)}}$

GROUP III. PART II.

$$\Delta =$$

80. $a(-\beta_a \sin \tfrac{1}{2}A + \beta_b \sin \tfrac{1}{2}B + \beta_c \sin \tfrac{1}{2}C)$
$= 2s(\beta_a \sin \tfrac{1}{2}A + \beta_b \sin \tfrac{1}{2}B + \beta_c \sin \tfrac{1}{2}C)$

81. $\sqrt{2r\beta_a\beta_b\beta_c\left[\dfrac{\cos(A-B)+\cos(B-C)+\cos(C-A)+1}{\cos A+\cos B+\cos C-1}\right]}$
$= \sqrt{2r_a\beta_a\beta_b\beta_c\left[\dfrac{\cos(A-B)+\cos(B-C)+\cos(C-A)+1}{-\cos A+\cos B+\cos C+1}\right]}$

82. $s^2 \tan \tfrac{1}{2}A \tan \tfrac{1}{2}B \tan \tfrac{1}{2}C = (s-a)^2 \tan \tfrac{1}{2}A \cot \tfrac{1}{2}B \cot \tfrac{1}{2}C$

83. $r^2 \cot \tfrac{1}{2}A \cot \tfrac{1}{2}B \cot \tfrac{1}{2}C = r_a^2 \cot \tfrac{1}{2}A \tan \tfrac{1}{2}B \tan \tfrac{1}{2}C.$

GROUP IV. PART I.

84. $a\dfrac{rr_a}{r_a-r} = a\dfrac{r_br_c}{r_b+r_c}$

85. $rr_a\dfrac{r_b+r_c}{a} = r_br_c\dfrac{r_a-r}{a}$

86. $(a+b)\dfrac{rr_c}{r+r_c} = (a-b)\dfrac{r_ar_b}{r_a-r_b}$

87. $rr_a\dfrac{r_b-r_c}{b-c} = r_br_c\dfrac{r+r_a}{b+c}$

88. $rr_a\sqrt{\dfrac{r_b+r_c}{r_a-r}} = r_br_c\sqrt{\dfrac{r_a-r}{r_b+r_c}}$

89. $rr_a\sqrt{\dfrac{4R-(r_a-r)}{r_a-r}} = r_br_c\sqrt{\dfrac{4R-(r_b+r_c)}{r_b+r_c}}$

90. $\dfrac{sh_ar_a}{h_a+2r_a} = \dfrac{(s-a)h_ar}{h_a-2r}$

91. $r\sqrt{\dfrac{h_ar_br_c}{h_a-2r}} = r_a\sqrt{\dfrac{h_ar_br_c}{h_a+2r_a}}$

92. $\sqrt{\dfrac{rr_a}{\left(\dfrac{1}{h_a}-\dfrac{1}{h_b}+\dfrac{1}{h}\right)\left(\dfrac{1}{h_a}+\dfrac{1}{h_b}-\dfrac{1}{h_c}\right)}}$
$= \sqrt{\dfrac{r_br_c}{\left(\dfrac{1}{h_a}+\dfrac{1}{h_b}+\dfrac{1}{h_c}\right)\left(-\dfrac{1}{h_a}+\dfrac{1}{h_b}+\dfrac{1}{h_c}\right)}}$

GROUP IV. PART II.

$\Delta =$

93. $rr_a \cot \tfrac{1}{2}A = r_b r_c \tan \tfrac{1}{2}A$

GROUP V.

94. $r^2 \cot \tfrac{1}{2}A + 2Rr \sin A = r_a^2 \cot \tfrac{1}{2}A - 2Rr_a \sin A = r_b^2 \cot \tfrac{1}{2}A - 2Rr_b \sin A$
$= r_c^2 \cot \tfrac{1}{2}A - 2Rr_c \sin A.$

MISCELLANEOUS EXPRESSIONS FOR THE AREA OF A PLANE TRIANGLE.

If we designate the distances from the orthocentre to the sides of the triangle by k_a, k_b, k_c and from the orthocentre to the vertices by k_a', k_b', k_c', then

95. $\Delta = \tfrac{1}{2}(ak_a + bk_b + ck_c)$

96.* $\Delta = \tfrac{1}{4}(ak_a' + bk_b' + ck_c')$

97.* $\Delta = \frac{1}{8R}(a^2k_a' \operatorname{cosec} A + b^2k_b' \operatorname{cosec} B + c^2k_c' \operatorname{cosec} C).$

If we designate the distances between the centres of the escribed circles by d_a, d_b, d_c, then

98.* $\Delta = \tfrac{1}{4}\sqrt{2d_ad_bd_cr \sin A \sin B \sin C}$

99. $\Delta = \frac{d_ad_bd_cr}{16R^2}$

100. $\Delta = \frac{d_ad_bd_c}{4R} \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C$

101.* $\Delta = r_ar_b\sqrt{\frac{d_c^2}{(r_a + r_b)^2} - 1}.$

If x_ay_a, x_by_b, x_cy_c designate the rectangular co-ordinates of a triangle in a plane, then

102. $\Delta = \tfrac{1}{2}[y_a(x_c - x_b) + y_b(x_a - x_c) + y_c(x_b - x_a)]$

and, in polar co-ordinates,

103. $\Delta = \tfrac{1}{2}[r_ar_b \sin(\theta'' - \theta') + r_br_c \sin(\theta''' - \theta'') + r_cr_a \sin(\theta' - \theta''')].$

If the equations of the sides of a triangle are

$$a_ax + b_ay + c_a = 0;\ a_bx + b_by + c_b = 0;\ a_cx + b_cy + c_c = 0,$$

then

104. $\Delta = \tfrac{1}{2}\left[\frac{[a_a(b_bc_c - b_cc_b) + a_b(b_cc_a - b_ac_c) + a_c(b_ac_b - b_bc_a)]^2}{(a_ab_b - b_aa_b)(a_bb_c - b_ba_c)(a_cb_a - b_ca_a)}\right].$

* *Tidsskrift for Mathematik.* 8°. Copenhagen, 1884, 5th Series, Vol. I, No. 2, pp. 78–79.
Mathesis. 8°. Gand, 1884, November, Vol. IV, p. 232.

If the co-ordinates of the vertices of a triangle are x_a, y_a, z_a; x_b, y_b, z_b; x_c, y_c, z_c, and

$$\alpha = x_a^2 + y_a^2 + z_a^2, \qquad \lambda = x_b x_c + y_b y_c + z_b z_c,$$
$$\beta = x_b^2 + y_b^2 + z_b^2, \qquad \mu = x_c x_a + y_c y_a + z_c z_a,$$
$$\gamma = x_c^2 + y_c^2 + z_c^2; \qquad \nu = x_a x_b + y_a y_b + z_a z_b; \text{ then}$$

105.* $$4\Delta^2 = -\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & \alpha & \nu & \mu \\ 1 & \nu & \beta & \lambda \\ 1 & \mu & \lambda & \gamma \end{vmatrix}$$

or, without determinants,

$$\Delta^2 = 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4,$$

where
$$a^2 = (x_b - x_c)^2 + (y_b - y_c)^2 + (z_b - z_c)^2,$$
$$b^2 = (x_c - x_a)^2 + (y_c - y_a)^2 + (z_c - z_a)^2,$$
$$c^2 = (x_a - x_b)^2 + (y_a - y_b)^2 + (z_a - z_b)^2,$$

and if $\alpha, \beta, \gamma, \lambda, \mu$ and ν, have the values set down above, then

$$a^2 = \beta + \gamma - 2\lambda,$$
$$b^2 = \gamma + \alpha - 2\mu,$$
$$c^2 = \alpha + \beta - 2\nu,$$

and
$$\Delta^2 = 2(\alpha + \beta - 2\nu)(\beta + \gamma - 2\lambda) - (\alpha + \beta - 2\nu)^2$$
$$+ 2(\beta + \gamma - 2\lambda)(\gamma + \alpha - 2\mu) - (\beta + \gamma - 2\lambda)^2$$
$$+ 2(\gamma + \alpha - 2\mu)(\alpha + \beta - 2\nu) - (\gamma + \alpha - 2\mu)^2,$$

which reduces to

$$\Delta^2 = 4(\lambda + \mu + \nu)^2 - 8(\alpha\lambda + \beta\mu + \gamma\nu) + 4(\alpha\beta + \beta\gamma + \gamma\alpha).$$

106. $\Delta = \frac{1}{8R}\left[(s-a)D_a^2 + (s-b)D_b^2 + (s-c)D_c^2 - sD^2\right]$ where D, D_a, D_b, D_c are the distances between the circumcentre and the in- and e-centres.

107. $\Delta = 4K$, where K is the area of the triangle formed by joining the feet of the medians.

**Giornale di Matematiche*, Naples, 1884, July, Vol. XXII, pp. 207–8.

108. $\Delta = \frac{L}{2\cos A \cos B \cos C}$, where L is the area of the pedal triangle or triangle formed by joining the feet of the perpendiculars.

109. $\Delta = M\frac{(a+b)(b+c)(c+a)}{2abc}$, where M is the area of the triangle formed by joining the feet of the internal bisectors.

110. $\Delta = \frac{2R}{r} \cdot N$, where N is the area of the triangle formed by joining the points of tangency of the in-circle.

CONCLUDING NOTE.—Owing to some changes and additions in the foregoing article made since its original preparation and while in press, the statement on page 135, Vol. I, that "the total number of formulæ * * * in this collection is *ninety-three*," etc., is no longer true.

The number of formulæ in the collection is shown below where the second column shows the number that result from counting as distinct those formulæ arising from permutation.

Group I,	31	31
Group II,	41	123
Group III,	11	44
Group IV,	10	60
Group V,	1	12
Miscellaneous Group,	16	18
Total,	110	288

[In the original manuscript all the forms arising from permuting the letters were given in full. To save space in printing, those forms which arose from mere cyclical permutation were omitted.—*O. S.*]

PROBLEM AND SOLUTION.

By DR. J. E. HENDRICKS, Des Moines, Iowa.

PROBLEM.—To inscribe a rectangle whose sides are c,x in another rectangle whose sides are a,b, and express x in functions of a, b, and c.