

Beta Functions in Past Papers

by TY WEBB

1945 Leaving Certificate Mathematics I Honours Q10iiib

$$\int_0^{\frac{\pi}{2}} \cos^3 x \, dx = \frac{1}{2} B\left(\frac{1}{2}, 2\right) = \frac{\Gamma(\frac{1}{2})\Gamma(2)}{2\Gamma(\frac{1}{2}+2)} = \frac{\sqrt{\pi} \times 1!}{2 \times \frac{4! \sqrt{\pi}}{4^2 \times 2!}} = \frac{2}{3}$$

1951 Leaving Certificate Mathematics I Honours Q11ii

$$\int_0^a \frac{x^{\frac{3}{2}}}{\sqrt{a-x}} \, dx = \int_0^a \frac{(ax)^{\frac{3}{2}}}{\sqrt{a-ax}} \, d(ax) = a^2 \int_0^1 x^{\frac{3}{2}} (1-x)^{-\frac{1}{2}} \, dx = a^2 B\left(\frac{5}{2}, \frac{1}{2}\right) = \frac{a^2 \Gamma(\frac{5}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{5}{2}+\frac{1}{2})} = \frac{a^2 \times \frac{4! \sqrt{\pi}}{4^2 \times 2!} \times \sqrt{\pi}}{2!} = \frac{3\pi a^2}{8}$$

1956 Leaving Certificate Mathematics I Honours Q8ii

$$\int_0^t x^2 (t-x)^{\frac{3}{2}} \, dx = \int_0^t (tx)^2 (t-tx)^{\frac{3}{2}} \, d(tx) = t^{\frac{9}{2}} \int_0^1 x^2 (1-x)^{\frac{3}{2}} \, dx = t^{\frac{9}{2}} B\left(3, \frac{5}{2}\right) = \frac{t^{\frac{9}{2}} \Gamma(3)\Gamma(\frac{5}{2})}{\Gamma(3+\frac{5}{2})} = \frac{t^{\frac{9}{2}} \times 2! \times \frac{4! \sqrt{\pi}}{4^2 \times 2!}}{\frac{10! \sqrt{\pi}}{4^5 \times 5!}} = \frac{16t^{\frac{9}{2}}}{315}$$

1962 Leaving Certificate Mathematics I Honours Q2ii

$$\int_0^{\frac{\pi}{2}} \sin^6 \theta \, d\theta = \frac{1}{2} B\left(\frac{7}{2}, \frac{1}{2}\right) = \frac{\Gamma(\frac{7}{2})\Gamma(\frac{1}{2})}{2\Gamma(\frac{7}{2}+\frac{1}{2})} = \frac{\frac{6! \sqrt{\pi}}{4^3 \times 3!} \times \sqrt{\pi}}{2 \times 3!} = \frac{5\pi}{32}$$

1980 HSC 4 unit Q7ia

$$\int_0^1 x(1-x)^{20} \, dx = B(2, 21) = \frac{\Gamma(2)\Gamma(21)}{\Gamma(2+21)} = \frac{1! \times 20!}{22!} = \frac{1}{462}$$

1980 HSC 4 unit Q7iib

$$\int_0^{\frac{\pi}{2}} \cos^2 x \, dx = \frac{1}{2} B\left(\frac{1}{2}, \frac{3}{2}\right) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}{2\Gamma(\frac{1}{2}+\frac{3}{2})} = \frac{\sqrt{\pi} \times \frac{2! \sqrt{\pi}}{4^1 \times 1!}}{2 \times 1!} = \frac{\pi}{4}$$

1985 HSC 4 unit Q1iii

$$\int_0^1 x(1-x)^{99} \, dx = B(2, 100) = \frac{\Gamma(2)\Gamma(100)}{\Gamma(2+100)} = \frac{1! \times 99!}{101!} = \frac{1}{10100}$$

1987 HSC 4 unit Q1iiib

$$\int_0^{\frac{\pi}{2}} \cos^5 x \, dx = \frac{1}{2} B\left(\frac{1}{2}, 3\right) = \frac{\Gamma(\frac{1}{2})\Gamma(3)}{2\Gamma(\frac{1}{2}+3)} = \frac{\sqrt{\pi} \times 2!}{2 \times \frac{6! \sqrt{\pi}}{4^3 \times 3!}} = \frac{8}{15}$$

1991 HSC 4 unit Q1ciii

$$\int_0^{\frac{\pi}{2}} \sin^4 x \, dx = \frac{1}{2} B\left(\frac{5}{2}, \frac{1}{2}\right) = \frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})}{2\Gamma(\frac{5}{2}+\frac{1}{2})} = \frac{\frac{4! \sqrt{\pi}}{4^2 \times 2!} \times \sqrt{\pi}}{2 \times 2!} = \frac{3\pi}{16}$$

1995 HSC 4 unit Q7a

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx &= \frac{1}{2} B\left(\frac{2n+1}{2}, \frac{1}{2}\right) = \frac{\Gamma(\frac{2n+1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\frac{2n+1}{2}+\frac{1}{2})} = \frac{\frac{(2n)! \sqrt{\pi}}{4^n n!} \times \sqrt{\pi}}{2n!} = \frac{(2n)! \pi}{2^{2n+1} (n!)^2} = \frac{\pi \binom{2n}{n}}{2^{2n+1}} \\ &= \frac{2n \times (2n-1) \times (2n-2) \times \dots \times 1}{(2n \times (2n-2) \times \dots \times 2)^2} \times \frac{\pi}{2} \text{ then even terms in numerator cancel} \\ &= \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot \dots \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \end{aligned}$$

and

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^{2n+1} x \, dx &= \frac{1}{2} B\left(n+1, \frac{1}{2}\right) = \frac{\Gamma(n+1)\Gamma(\frac{1}{2})}{2\Gamma(n+1+\frac{1}{2})} = \frac{n!\sqrt{\pi}}{\frac{2(2n+2)!\sqrt{\pi}}{4^{n+1}(n+1)!}} = \frac{2^{2n}(n!)^2}{(2n+1)!} = \frac{2^{2n}}{(2n+1)\binom{2n}{n}} \\ &= \frac{((2n) \times (2n-2) \times \dots \times 2)^2}{(2n+1) \times (2n) \times \dots \times 1} \text{ then even terms in denominator cancel} \\ &= \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdot \dots \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1\end{aligned}$$

2020 HSC Extension 2 Q16biii

$$\int_0^1 x^n(1-x)^n \, dx = B(n+1, n+1) = \frac{\Gamma(n+1)\Gamma(n+1)}{\Gamma(n+1+n+1)} = \frac{(n!)^2}{(2n+1)!}$$

2023 HSC Extension 2 Q15aii

$$\text{From 1995 Q7a, } \int_0^{\frac{\pi}{2}} \sin^{2n+1} x \, dx = \frac{1}{2} B\left(n+1, \frac{1}{2}\right) = \frac{2^{2n}(n!)^2}{(2n+1)!}$$

$$\text{and from 2020 Q16biii, } \int_0^1 x^n(1-x)^n \, dx = B(n+1, n+1) = \frac{(n!)^2}{(2n+1)!}$$

$$\text{Hence } \int_0^1 x^n(1-x)^n \, dx = \frac{1}{2^{2n}} \int_0^{\frac{\pi}{2}} \sin^{2n+1} x \, dx$$