Carlingford High School
Mathematics Extension 1
Higher School Certificate
Trial Examination 2024

NESA Number:

Circle teacher: Bennett Cheng Diep Strilakos

Mathematics Extension 1

General Instructions

- Reading time: 10 minutes
- Working time: 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- In Section II, start each question in a new booklet. Use the Question 12 Writing Booklet for Question 12.
- Show relevant mathematical reasoning and/or calculations
- Write your NESA number on each writing booklet

Total Marks: 70

Section I – 10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 60 marks

- Attempt Questions 11 – 14
- Allow about 1 hour and 45 minutes for this section

	MC	Q11	Q12	Q13	Q14	Total
Further work with functions	4, 9 **/2**	f **/4**	a, e **/6**			**/12**
Further Trigonometry and equations	1, 3 **/2**	a **/2**	b **/4**	a **/3**		**/11**
Combinatorics	2, 6 **/2**	c, e **/4**	d **/2**	c **/3**		**/11**
Vectors	5, 8 **/2**	d **/3**			c **/8**	**/13**
Further Calculus Skills	7, 10 **/2**	b **/2**	c **/3**	b, d **/9**	a **/4**	**/20**
Proof (MI)					b **/3**	**/3**
Total	/10	/15	/15	/15	/15	/70

Section I (10 marks)
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the Multiple Choice Answer Sheet for Questions 1 – 10

1 Determine which inverse trigonometric function below could be described as an odd function with a domain of [–3, 3]?

(A) $y = 2\, cos^{-1}\left(\frac{x}{3}\right)$

(B) $y = 2\, sin^{-1}\left(\frac{x}{3}\right)$

(C) $y = 3\, sin^{-1}\left(\frac{x}{2}\right)$

(D) $y = 2\, tan^{-1}\left(\frac{x}{3}\right)$

2 Consider the three statements below:

Statement 1: In a group of thirteen people, at least two will have their birthday in the same month.

Statement 2: A room which has five windows in its four walls must have at least one wall which has at least two windows.

Statement 3: A drawer contains ten socks which are identical except that they are a mixture of seven different colours. If six socks are drawn randomly from the drawer there must be at least one pair of socks of matching colours.

Which statements are correct applications of the Pigeonhole Principle?

(A) Statements 1 and 2 only

(B) Statements 1 and 3 only

(C) Statements 2 and 3 only

(D) All three statements

3 What is the minimum value of $f(x) = \sqrt{3}\, sin\, x + cos\, x + 5$?

(A) 2

(B) 3

(C) 4

(D) 7

4 What is the Cartesian equation of the curve with the parametric equations below?

$$x = 3sin\theta + 1$$
$$y = 3cos\theta$$

(A) $x^2 + (y-1)^2 = 3$

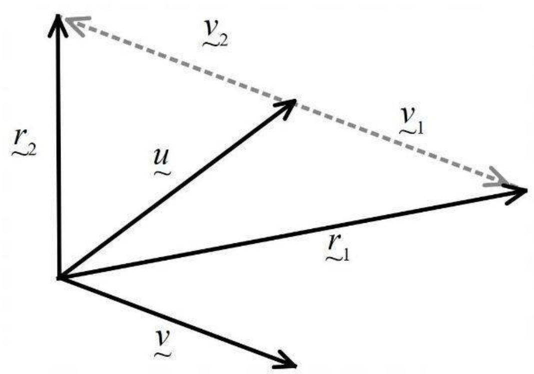
(B) $(x-1)^2 + y^2 = 3$

(C) $x^2 + (y-1)^2 = 9$

(D) $(x-1)^2 + y^2 = 9$

5 The diagram shows two vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

Two resultant vectors, $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$, are constructed using $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$ which are parallel to, and equal in length to $\underset{\sim}{v}$.



Which statement is true?

(A) $\underset{\sim}{r}_1 = \underset{\sim}{u} - \underset{\sim}{v}$ and $\underset{\sim}{r}_2 = \underset{\sim}{u} + \underset{\sim}{v}$

(B) $\underset{\sim}{r}_1 = \underset{\sim}{v} + \underset{\sim}{u}$ and $\underset{\sim}{r}_2 = \underset{\sim}{v} - \underset{\sim}{u}$

(C) $\underset{\sim}{r}_1 = \underset{\sim}{u} + \underset{\sim}{v}$ and $\underset{\sim}{r}_2 = \underset{\sim}{u} - \underset{\sim}{v}$

(D) $\underset{\sim}{r}_1 = \underset{\sim}{v} - \underset{\sim}{u}$ and $\underset{\sim}{r}_2 = \underset{\sim}{u} - \underset{\sim}{v}$

6 How many distinct arrangements are possible from the letters of the word NECESSITIES?

(A) 50 400

(B) 554 400

(C) 1 663 200

(D) 39 916 800

7 If $y = \tan^{-1}\left(\frac{1}{x}\right)$, which of the following is an expression for $\frac{dy}{dx}$?

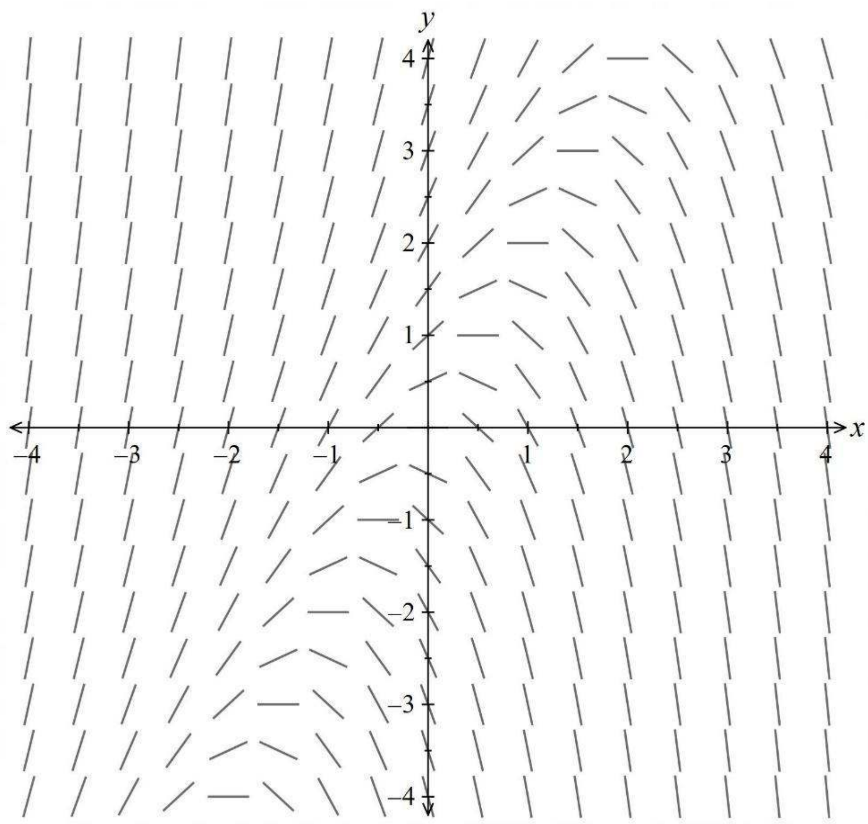
(A) $\frac{dy}{dx} = -\tan^2 y$

(B) $\frac{dy}{dx} = -\cot^2 y$

(C) $\frac{dy}{dx} = -\cos^2 y$

(D) $\frac{dy}{dx} = -\sin^2 y$

8 The direction field for a differential equation is shown below.



Which of the differential equations could give this direction field?

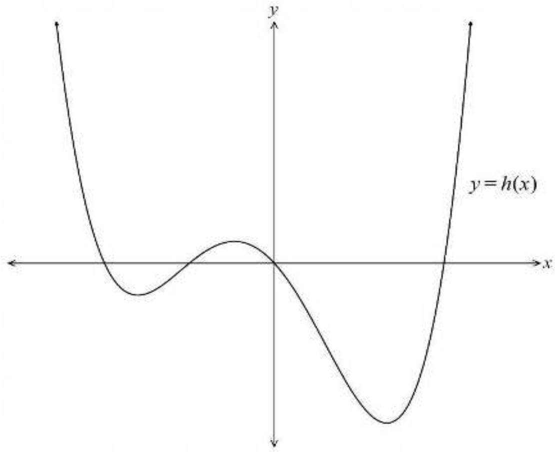
(A) $y' = y - 2x$

(B) $y' = y + 2x$

(C) $y' = y + x$

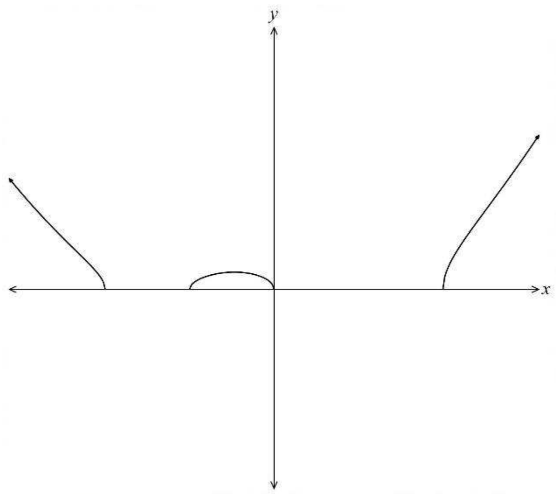
(D) $y' = x - y$

9 The graph of $y = h(x)$ is given below.

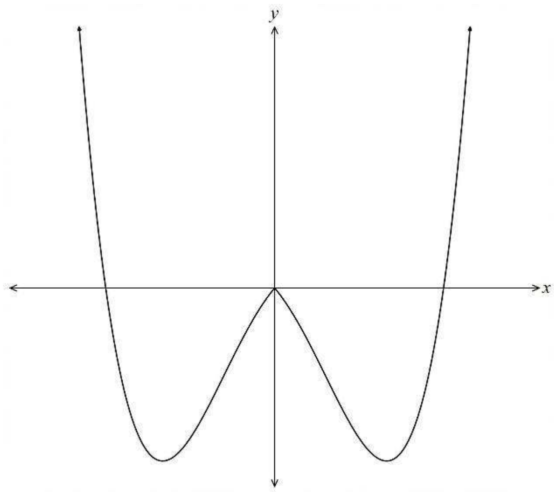


Which of the following shows the graph of $y = |h(x)|$?

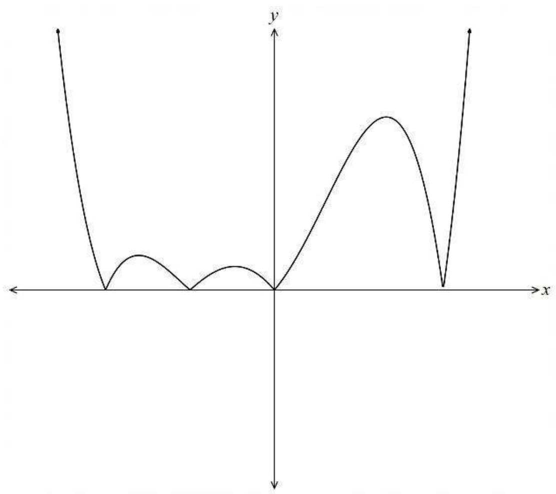
A.



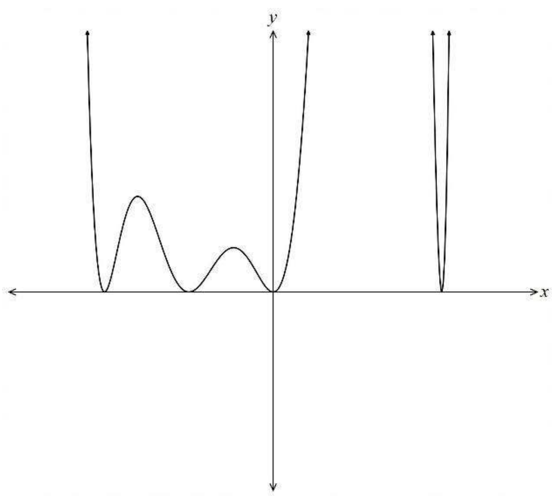
B.



C.



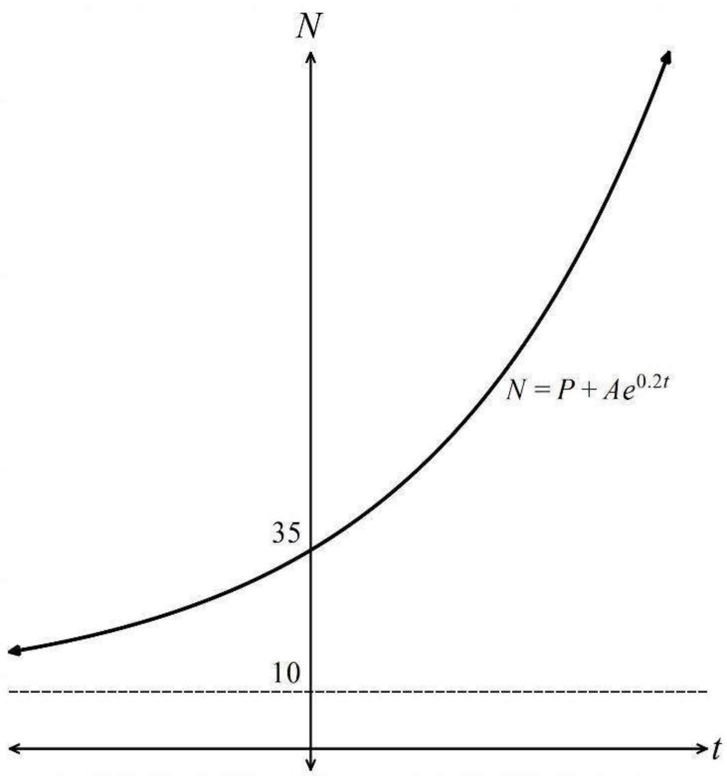
D.



10 The quantity N increases exponentially over time t according to the equation

$$N = P + A\,e^{0.2t}.$$

The graph below shows the change in N over time.



Which equation describes the rate of change of N with respect to t?

(A) $\frac{dN}{dt} = 0.2(N - 10)$

(B) $\frac{dN}{dt} = 0.2(N - 25)$

(C) $\frac{dN}{dt} = 10(N - 0.2)$

(D) $\frac{dN}{dt} = 25(N - 10)$

Section II (60 marks)
Attempt Questions 11 – 14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

Question 11 (15 marks)
Use a new writing booklet

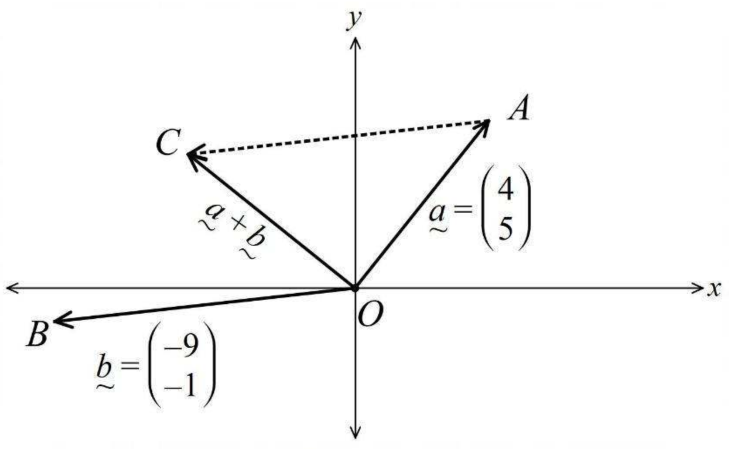
(a) Show that $(2 - \sec^2 A)\tan(2A) = 2\tan A$. **2**

(b) Find the exact value of the integral below: **2**

$$\int_0^1 \frac{2x^2+1}{25+(2x^3+3x)^2}\,dx$$

(c) Find the coefficient of x^9 in the expansion of $\left(\frac{3x^3}{2} + \frac{1}{5}\right)^6$. **2**

(d) The diagram shows the vectors $\underset{\sim}{a} = \begin{bmatrix}4\\5\end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix}-9\\-1\end{bmatrix}$ and the resultant vector $\underset{\sim}{a} + \underset{\sim}{b}$.



(i) By using vector methods, show that ΔOAC is an isosceles triangle. **2**

(ii) Find the size of $\angle OAC$. **1**

Question 11 continues on Page 9

(e) Four boys, including Mark, and six girls arrange themselves in a line. **2**

How many arrangements are possible if Mark must have a girl next to him on either side?

(f) Consider the polynomial $P(x) = x^4 - 9x^3 + 25x^2 - 27x + 10$.

(i) Show that $x = 1$ is a double zero of $P(x)$. **2**

(ii) Hence, or otherwise, factorise $P(x)$ into linear factors. **2**

End of Question 11

Question 12 (15 marks)
Use the **Question 12 Writing Booklet**

(a) (i) A function is defined as $f(x) = (x-1)^3 - 4$ for $x \geq 0$. **2**

In the Question 12 Writing Booklet, the graph of $y = f(x)$ is shown.

On the same diagram, sketch the graph of $y = f^{-1}(x)$. Clearly show the axis of symmetry.

(ii) Write the equation that describes $f^{-1}(x)$ and state its domain. **2**

(b) (i) Use the t-formulae to show that **2**

$$\sin x - \tan\frac{x}{2} = \tan\frac{x}{2}\cos x.$$

(ii) Hence, or otherwise, solve the equation **2**

$$\sin x - \tan\frac{x}{2} = 0 \text{ for } 0 \leq x \leq 2\pi.$$

(c) Use the method of separation of variables to solve the differential equation $\frac{dy}{dx} = \frac{3y}{x}$, **3**

given that $\frac{dy}{dx} = 6$ when $x = 2$.

(d) Ten people including Bob and Mary are seated randomly around a circular table. **2**

What is the probability that Bob and Mary do not sit next to each other?

(e) The polynomial equation $x^3 - 5x^2 + 2x - 12 = 0$ has roots α, β and γ. **2**

Find the equation of the polynomial with roots $2\alpha,\ 2\beta$ and 2γ.

End of Question 12

Question 13 (15 marks)
Use a new writing booklet

(a) (i) Prove **1**

$$\sin x + \sin 2x + \sin 3x = \sin 2x\,(2\cos x + 1).$$

(ii) Hence, solve $\sin x + \sin 2x + \sin 3x = 0$ for $x \in \left[0, \frac{\pi}{2}\right]$. **2**

(b) Find, in simplest exact form, the coordinates of the stationary point(s) on the curve **3**

$$y = x^2 + \cos^{-1} x.$$

(c) (i) Prove the combinatorial identity: **1**

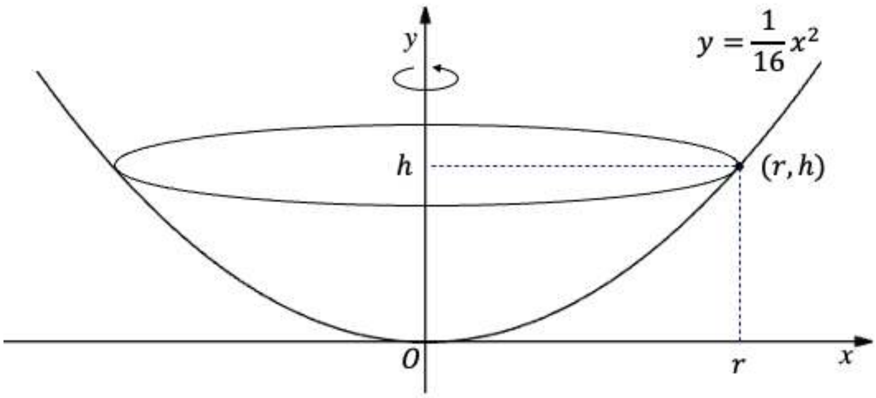
$${}^{n+1}C_{r+1} = {}^{n}C_r + {}^{n}C_{r+1}$$

(ii) Given $p = {}^{n}C_4$ and $q = {}^{n}C_5$, show that: **2**

$${}^{n}C_{n-4} + {}^{n+1}C_{n-5} = p + \frac{q(n+1)}{6}$$

Question 13 continues on Page 12

(d) The shape of a water reservoir is created by rotating the parabola $y = \frac{1}{16}x^2$ about the y-axis between $y = 0$ and $y = h$.



The variable h (metres) represents the depth of the water in the reservoir at time t (hours), and r (metres) represents the radius of the circular area (A) of the water surface at that time.

(i) Show that the volume of the water is given by $V = \frac{A^2}{32\pi}$. **2**

(ii) The area A is decreasing at a constant rate of $\frac{\pi}{20} m^2/h$. **2**

Show that the differential equation below represents the rate at which the water volume is decreasing:

$$\frac{dV}{dt} = -\frac{1}{80}\sqrt{2\pi V}$$

(iii) Initially, the volume of the water is $20\ 000\ m^3$. **2**

By solving the differential equation, find how long it takes for the volume of water to fall below $10\ 000\ m^3$. Express your answer correct to the nearest hour.

End of Question 13

Question 14 (15 marks)
Use a new writing booklet

(a) Find the value of $\int_1^{49} \frac{1}{\sqrt{x}\sqrt{1+\sqrt{x}}} dx$ using the substitution $u = 1 + \sqrt{x}$. **4**

Express your answer in the form $a\sqrt{b}$ where a and b are both positive integers.

(b) Use Mathematical Induction to prove, for all integers $n \geq 1$, that **3**

$$\left(1 \times 1! + \frac{0}{1!}\right) + \left(2 \times 2! + \frac{1}{2!}\right) + \left(3 \times 3! + \frac{2}{3!}\right) + \cdots + \left(n \times n! + \frac{n-1}{n!}\right) = (n+1)! - \frac{1}{n!}.$$

(c) A particle is projected from a point O on the horizontal plane with velocity V and at an angle θ to the horizontal.

(i) Find H, the greatest height reached in terms of V, θ and g, where g is the acceleration due to gravity. Show necessary working. **2**

(ii) A point A on the trajectory is at height h above the plane and is such that the particle passes through A before reaching its highest point. Show that the time taken to travel from O to A is **2**

$$\sqrt{\frac{2H}{g}} - \sqrt{\frac{2(H-h)}{g}}.$$

(iii) B is the other point on the trajectory which is at the height h above the plane. If the time taken to travel from A to B is equal to the time taken to travel from O to the highest point, show that **2**

$$h = \frac{3}{4}H\,.$$

(iv) Hence, show that the gradient of the trajectory at A is given by **2**

$$\frac{dy}{dx} = \frac{1}{2} tan\, \theta\,.$$

End of paper

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2024 Trial HSC Examination
Mathematics Extension 1 Course

NESA Number

Teacher ______________________

Section I – Multiple Choice Answer Sheet

Allow about 15 minutes for this section

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample:	2 + 4 =	(A) 2	(B) 6	(C) 8	(D) 9
		A ○	B ●	C ○	D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

		A ●	B ●	C ○	D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word **correct** and drawing an arrow as follows.

		A ●	B ● correct	C ○	D ○

1. A ○ B ○ C ○ D ○
2. A ○ B ○ C ○ D ○
3. A ○ B ○ C ○ D ○
4. A ○ B ○ C ○ D ○
5. A ○ B ○ C ○ D ○
6. A ○ B ○ C ○ D ○
7. A ○ B ○ C ○ D ○
8. A ○ B ○ C ○ D ○
9. A ○ B ○ C ○ D ○
10. A ○ B ○ C ○ D ○

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NESA Number

Carlingford High School

2024

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

Writing Booklet

Question 12

Instructions

- Use this booklet for Question 12 ⇒ 12
- Write the number of this booklet and the number of booklets that you have used for this question (e.g. 1 of 2)

of

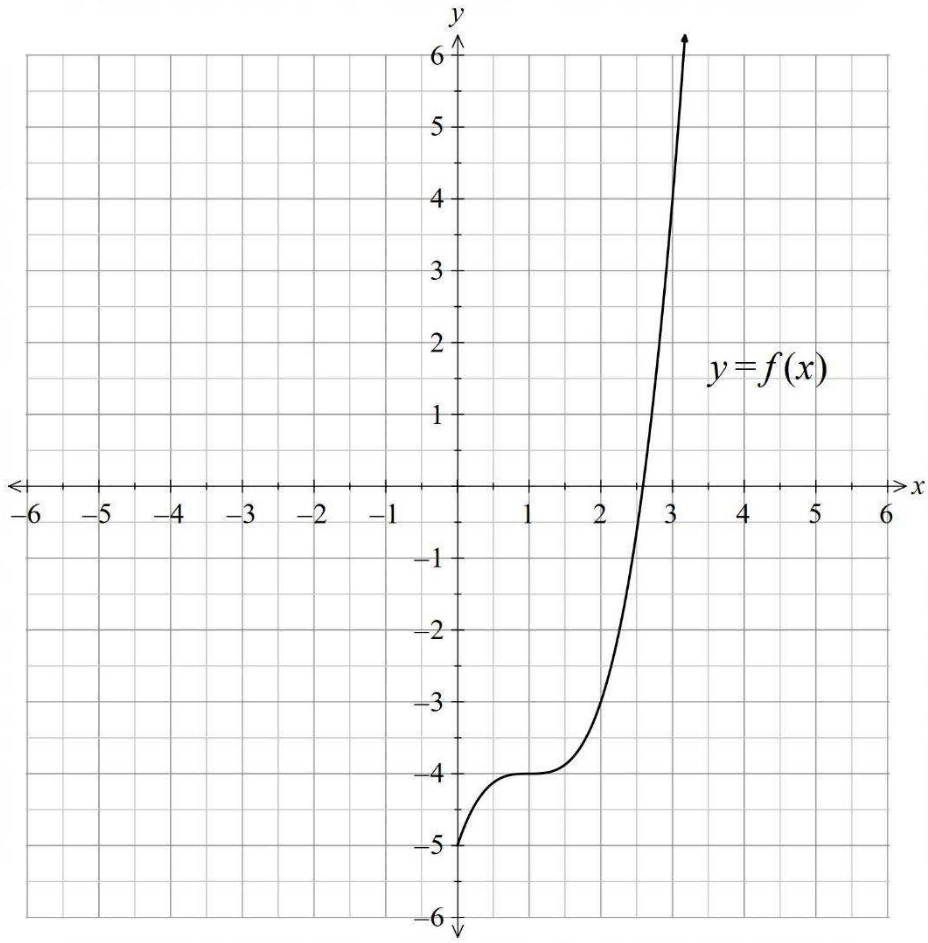
This booklet | Number of booklets for this question

- Write your NESA Number at the top of this page
- If you have not attempted the question(s), you must still hand in the writing booklet with "NOT ATTEMPTED" written clearly on the front cover.

Start here for Question 12

(a) (i) The graph of $f(x) = (x-1)^3 - 4$ for $x \geq 0$ is shown below.

On the same diagram, sketch the graph of $y = f^{-1}(x)$. Clearly show the axis of symmetry.



Tick this box if you have continued this answer in another writing booklet.

Carlingford High School
Mathematics Extension 1
Higher School Certificate
Trial Examination 2024

NESA Number: SOLUTIONS

Circle teacher: Bennett Cheng Diep Strilakos

Mathematics Extension 1

General Instructions

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- Working time: 2 hours
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- Show relevant mathematical reasoning and/or calculations
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Total Marks: 70

Section I – 10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 60 marks

- Attempt Questions 11 – 14
- Allow about 1 hour and 45 minutes for this section

	MC	Q11	Q12	Q13	Q14	Total
Further work with functions	4, 9 **/2**	f **/4**	a, e **/6**			**/12**
Further Trigonometry and equations	1, 3 **/2**	a **/2**	b **/4**	a **/3**		**/11**
Combinatorics	2, 6 **/2**	c, e **/4**	d **/2**	c **/3**		**/11**
Vectors	5, 8 **/2**	d **/3**			c **/8**	**/13**
Further Calculus Skills	7, 10 **/2**	b **/2**	c **/3**	b, d **/9**	a **/4**	**/20**
Proof (MI)					b **/3**	**/3**
Total	/10	/15	/15	/15	/15	/70

Section I (10 marks)
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the Multiple Choice Answer Sheet for Questions 1 – 10

1 Determine which inverse trigonometric function below could be described as an odd function with a domain of [–3, 3]?

(A) $y = 2\,cos^{-1}\left(\frac{x}{3}\right)$

(B) $y = 2\,sin^{-1}\left(\frac{x}{3}\right)$

(C) $y = 3\,sin^{-1}\left(\frac{x}{2}\right)$

(D) $y = 2\,tan^{-1}\left(\frac{x}{3}\right)$

A is neither an odd nor even function.

B is an odd function with a domain of [–3, 3].

C is an odd function with a domain of [–2, 2].

D is an odd function with a domain of $(-\infty, \infty)$.

2 Consider the three statements below:

Statement 1: In a group of thirteen people, at least two will have their birthday in the same month.

Statement 2: A room which has five windows in its four walls must have at least one wall which has at least two windows.

Statement 3: A drawer contains ten socks which are identical except that they are a mixture of seven different colours. If six socks are drawn randomly from the drawer there must be at least one pair of socks of matching colours.

Which statements are correct applications of the Pigeonhole Principle?

(A) Statements 1 and 2 only
(B) Statements 1 and 3 only
(C) Statements 2 and 3 only
(D) All three statements

Consider the three statements in light of the pigeonhole principle which states:
If there are n pigeonholes and n+1 pigeons to go into them, then at least one pigeonhole must hold 2 or more pigeons.

Statement 1 There are 12 months and 13 people's birth-months to be matched with them, so at least one month must have at least two people. This is a correct application of the pigeonhole principle.

Statement 2 There are 4 walls and 5 windows to be placed in them, so at least one wall must have at least two windows. This is a correct application of the pigeonhole principle.

Statement 3 There are 7 colours and 10 socks to be placed in them, but only 6 are drawn, so all 6 could be different colours. To work correctly 8 socks would need to be drawn $(7 + 1)$. This is not a correct application of the pigeonhole principle.

Statements 1 and 2 only are correct applications of the pigeonhole principle.

3 What is the minimum value of $f(x) = \sqrt{3}\, sin\, x + cos\, x + 5$?

(A) 2

(B) 3

(C) 4

(D) 7

$$\begin{aligned} f(x) &= \sqrt{3}\, sin\, x + cos\, x + 5 \\ &= 2\left(\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x\right) + 5 \\ &= 2\left(\cos\frac{\pi}{6}\sin x + \sin\frac{\pi}{6}\cos x\right) + 5 \\ &= 2\sin\left(x + \frac{\pi}{6}\right) + 5 \\ f_{min} &= -2 + 5 \\ &= 3 \end{aligned}$$

4 What is the Cartesian equation of the curve with the parametric equations below?

$$x = 3sin\theta\ + 1$$
$$y = 3\, cos\, \theta$$

(A) $x^2 + (y-1)^2 = 3$

(B) $(x-1)^2 + y^2 = 3$

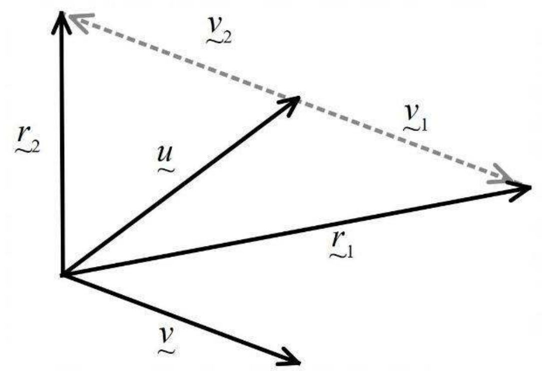
(C) $x^2 + (y-1)^2 = 9$

(D) $(x-1)^2 + y^2 = 9$

$$\begin{cases} x = 3sin\theta\ + 1 \\ y = 3\, cos\, \theta \end{cases}$$
$$\begin{cases} x - 1 = 3\, sin\theta \\ \quad y = 3\, cos\theta \end{cases}$$
$$\begin{cases} (x-1)^2 = 3^2\sin^2\theta \\ \quad y^2 = 3^2\cos^2\theta \end{cases}$$
$$(x-1)^2 + y^2 = 3^2(\sin^2\theta + \cos^2\theta)$$
$$(x-1)^2 + y^2 = 9$$

5 The diagram shows two vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

Two resultant vectors, $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$, are constructed using $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$ which are parallel to, and equal in length to $\underset{\sim}{v}$.



Which statement is true?

(A) $\underset{\sim}{r}_1 = \underset{\sim}{u} - \underset{\sim}{v}$ and $\underset{\sim}{r}_2 = \underset{\sim}{u} + \underset{\sim}{v}$

(B) $\underset{\sim}{r}_1 = \underset{\sim}{v} + \underset{\sim}{u}$ and $\underset{\sim}{r}_2 = \underset{\sim}{v} - \underset{\sim}{u}$

(C) $\underset{\sim}{r}_1 = \underset{\sim}{u} + \underset{\sim}{v}$ and $\underset{\sim}{r}_2 = \underset{\sim}{u} - \underset{\sim}{v}$

(D) $\underset{\sim}{r}_1 = \underset{\sim}{v} - \underset{\sim}{u}$ and $\underset{\sim}{r}_2 = \underset{\sim}{u} - \underset{\sim}{v}$

$$\begin{aligned} \underset{\sim}{r}_1 &= \underset{\sim}{u} + \underset{\sim}{v}_1 \\ &= \underset{\sim}{u} + \underset{\sim}{v} \end{aligned}$$

$$\begin{aligned} \underset{\sim}{r}_2 &= \underset{\sim}{u} + \underset{\sim}{v}_2 \\ &= \underset{\sim}{u} - \underset{\sim}{v} \end{aligned}$$

6 How many distinct arrangements are possible from the letters of the word NECESSITIES?

(A) 50 400

(B) 554 400

(C) 1 663 200

(D) 39 916 800

NECESSITIES has 11 letters with E and S occurring 3 times, and I occurring twice.

Distinct arrangements $= \frac{11!}{3!\times 3!\times 2!}$

$= 554\ 400$

7 If $y = \tan^{-1}\left(\frac{1}{x}\right)$, which of the following is an expression for $\frac{dy}{dx}$?

(A) $\frac{dy}{dx} = -\tan^2 y$

(B) $\frac{dy}{dx} = -\cot^2 y$

(C) $\frac{dy}{dx} = -\cos^2 y$

(D) $\frac{dy}{dx} = -\sin^2 y$

<u>Method 1:</u>

$y = \tan^{-1}\left(\frac{1}{x}\right)$

$\tan y = \frac{1}{x}$

$x = (\tan y)^{-1}$

$\frac{dx}{dy} = -(\tan y)^{-2}\sec^2 y$

$= -\frac{\cos^2 y}{\sin^2 y} \times \frac{1}{\cos^2 y}$

$\frac{dy}{dx} = -\sin^2 x$

<u>Method 2:</u>

$y = \tan^{-1}\left(\frac{1}{x}\right)$

$\frac{dy}{dx} = \frac{-\frac{1}{x^2}}{1+\frac{1}{x^2}}$ where $\frac{1}{x} = \tan y$

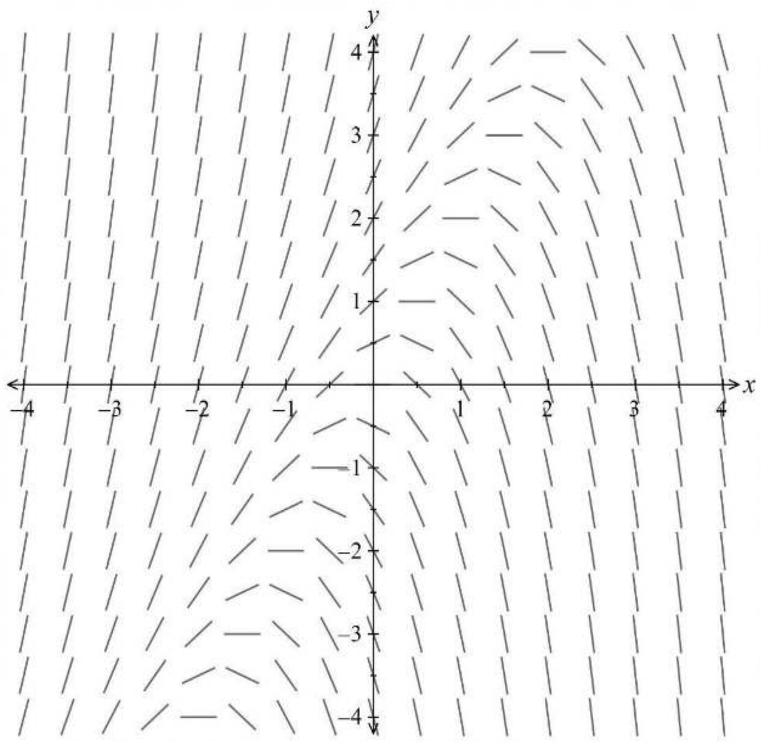
$= -\frac{\tan^2 y}{1+\tan^2 y}$

$= -\frac{\tan^2 y}{\sec^2 y}$

$= -\frac{\sin^2 y}{\cos^2 y} \times \cos^2 y$

$= -\sin^2 y$

8 The direction field for a differential equation is shown below.



Which of the differential equations could give this direction field?

Take the point (1, 1) (which has a negative gradient in the direction field) and test with the four equations.

(A) $y' = y - 2x$ $\quad y' = 1 - 2(1) = -1$

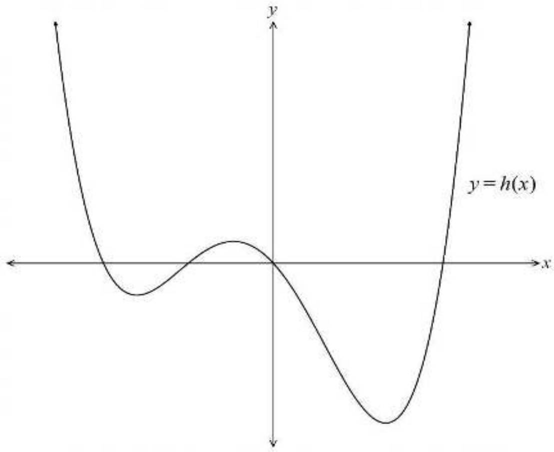
(B) $y' = y + 2x$ $\quad y' = 1 + 2(1) = 3$

(C) $y' = y + x$ $\quad y' = 1 + 1 = 2$

(D) $y' = x - y$ $\quad y' = 1 - 1 = 0$

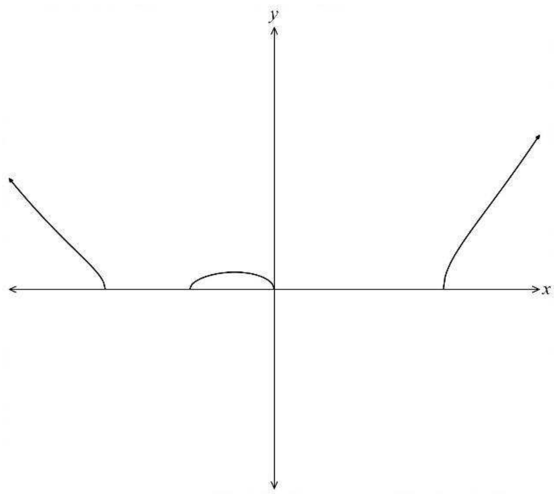
Only A has a negative gradient at the point (1, 1).

9 The graph of $y = h(x)$ is given below.

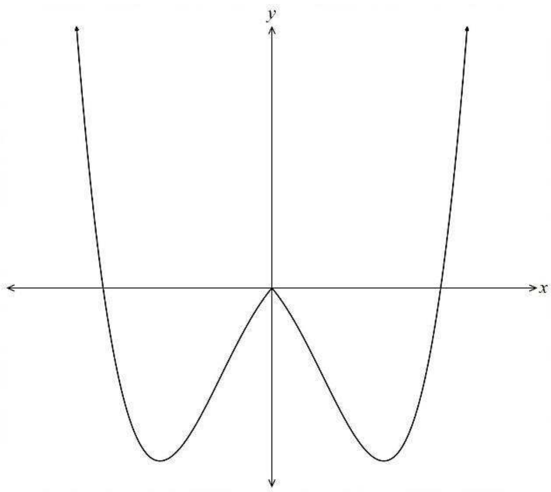


Which of the following shows the graph of $y = |h(x)|$?

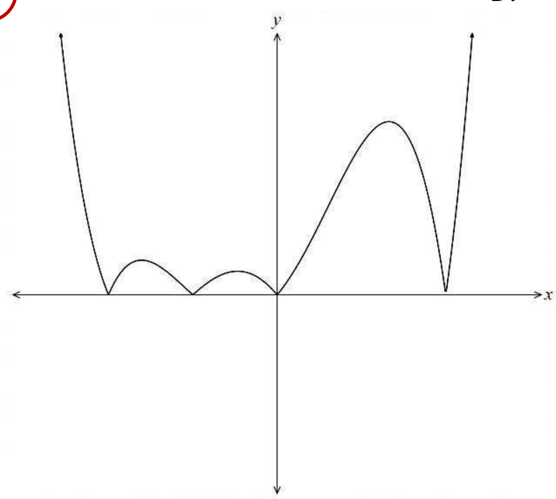
A.



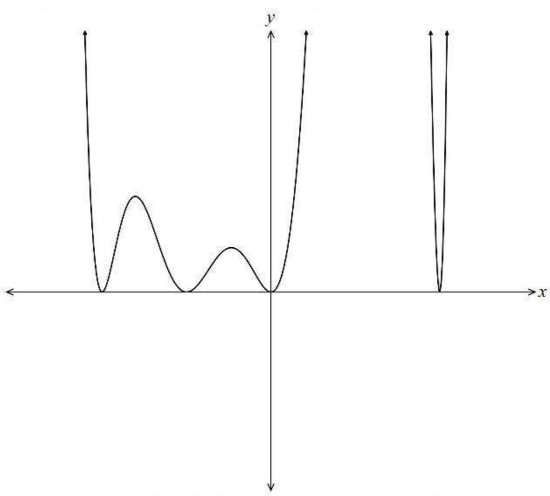
B.



C.



D.

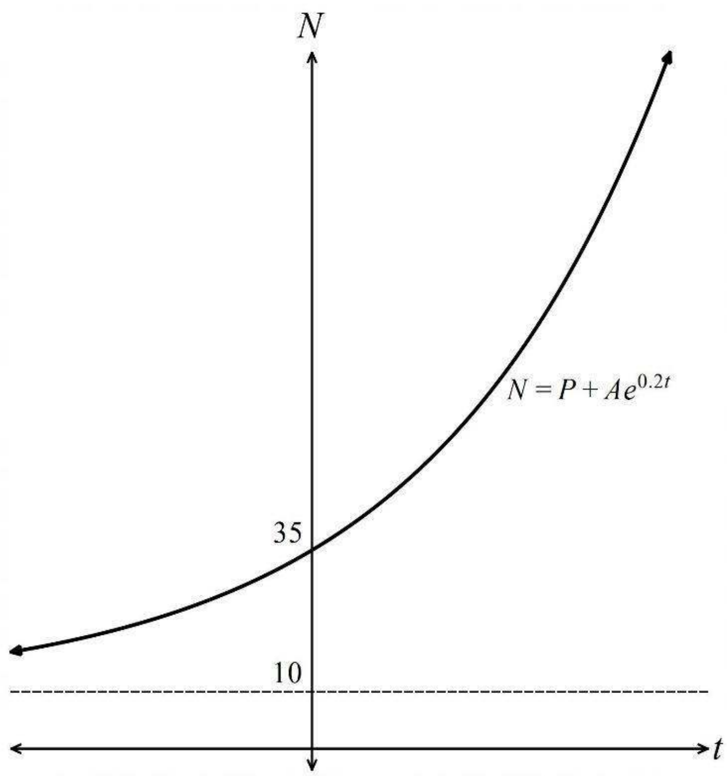


The sections where the curve dips below the x-axis are reflected in the x-axis (the negative y-values become positive when the absolute value is taken), while those above the x-axis remain unchanged.

10 The quantity N increases exponentially over time t according to the equation

$$N = P + A\,e^{0.2t}.$$

The graph below shows the change in N over time.



Which equation describes the rate of change of N with respect to t?

(A) $\frac{dN}{dt} = 0.2(N - 10)$

(B) $\frac{dN}{dt} = 0.2(N - 25)$

(C) $\frac{dN}{dt} = 10(N - 0.2)$

(D) $\frac{dN}{dt} = 25(N - 10)$

When $t = 0, N = 35$.
$35 = P + A$
When $t \to -\infty, N \to 10$.
$\therefore P = 10$
$N = 10 + 25e^{0.2t}$
$\frac{dN}{dt} = 0.2 \times 25e^{0.2t}$
$= 0.2(N - 10)$

Section II (60 marks)
Attempt Questions 11 – 14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

Question 11 (15 marks)
Use a new writing booklet

(a) Show that $(2 - \sec^2 A)\tan(2A) = 2\tan A$. **2**

$$LHS = [2 - (1 + \tan^2 A)] \times \frac{2\tan A}{1-\tan^2 A} \quad 1$$
$$= (1 - \tan^2 A)\frac{2\tan A}{1-\tan^2 A}$$
$$= 2\tan A \quad 1$$
$$= RHS$$

(b) Find the exact value of the integral below: **2**

$$\int_0^1 \frac{2x^2+1}{25+(2x^3+3x)^2}\,dx$$

$$\int_0^1 \frac{2x^2+1}{25+(2x^3+3x)^2}\,dx = \frac{1}{3}\int_0^1 \frac{6x^2+3}{5^2+(2x^3+3x)^2}\,dx$$
$$= \frac{1}{3}\left[\frac{1}{5}\tan^{-1}\left(\frac{2x^3+3x}{5}\right)\right]_0^1 \quad 1$$
$$= \frac{1}{15}(\tan^{-1}1 - \tan^{-1}0)$$
$$= \frac{1}{15}\left(\frac{\pi}{4} - 0\right)$$
$$= \frac{\pi}{60} \quad 1$$

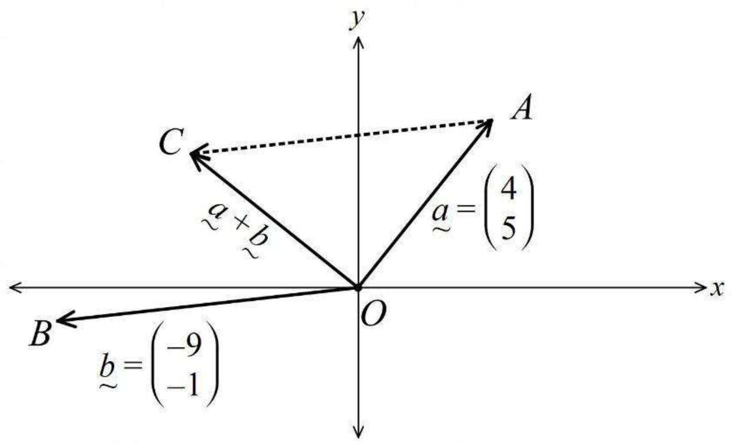
(c) Find the coefficient of x^9 in the expansion of $\left(\frac{3x^3}{2} + \frac{1}{5}\right)^6$. **2**

x^9 will come from $(x^3)^3$.

$$^6C_3\left(\frac{3x^3}{2}\right)^3\left(\frac{1}{5}\right)^3 = 20\left(\frac{27x^3}{8}\right)\left(\frac{1}{125}\right) \quad 1$$
$$= \frac{27}{50}x^3$$

The coefficient of x^9 is $\frac{27}{50}$. 1

(d) The diagram shows the vectors $\underset{\sim}{a} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix} -9 \\ -1 \end{bmatrix}$ and the resultant vector $\underset{\sim}{a} + \underset{\sim}{b}$.



(i) By using vector methods, show that ΔOAC is an isosceles triangle. **2**

$$\overrightarrow{OC} = \underset{\sim}{a} + \underset{\sim}{b}$$
$$= \begin{bmatrix} 4-9 \\ 5-1 \end{bmatrix}$$
$$= \begin{bmatrix} -5 \\ 4 \end{bmatrix}$$
$$\left|\overrightarrow{OC}\right| = \sqrt{(-5)^2 + 4^2}$$
$$= \sqrt{41} \qquad 1$$
$$\left|\overrightarrow{OA}\right| = |\underset{\sim}{a}|$$
$$= \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$
$$= \sqrt{4^2 + 5^2}$$
$$= \sqrt{41}$$

$\left|\overrightarrow{OC}\right| = \left|\overrightarrow{OA}\right|$ (two sides of ΔOAC equal) 1

$\therefore$ ΔOAC is an isosceles triangle.

(ii) Find the size of $\angle OAC$. **1**

$$\overrightarrow{OA} \cdot \overrightarrow{OC} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ 4 \end{bmatrix}$$
$$= -20 + 20$$
$$= 0$$

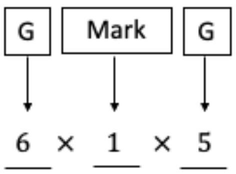
$\therefore \angle AOC = 90°$

$\therefore$ $\angle OAC = \angle OCA = \frac{90°}{2} = 45°$ 1

(e) Four boys, including Mark, and six girls arrange themselves in a line. **2**

How many arrangements are possible if Mark must have a girl next to him on either side?

<u>Method 1:</u>



Step 1: Choose one of the girls to place on one side of Mark: 6 ways
Step 2: Choose another girl to place on the other side of Mark: 5 ways
Step 3: Mark and the two girls together as 1 entity. Plus 7 others.
That makes 8 entities in total, arranged in 8! ways.
Total number arrangements $= 6 \times 5 \times 8!$
$= 1\ 209\ 600$

<u>Method 2:</u>

Step 1: Choose two girls and arrange them on either side of Mark: ${}^6C_2 \times 2!$ (i.e. ${}^6P_2$)
$= 30$ ways 1

Step 2: Mark and the two girls together as 1 entity. Plus 7 others.
That makes 8 entities in total, arranged in 8! ways.
Total number arrangements $= 30 \times 8!$ 1
$= 1\ 209\ 600$

(f) Consider the polynomial $P(x) = x^4 - 9x^3 + 25x^2 - 27x + 10$.

(i) Show that $x = 1$ is a double zero of $P(x)$. **2**

$P(1) = 1 - 9 + 25 - 27 + 10$
$= 0$ 1
$\therefore x = 1$ is a zero of $P(x)$.

$P'(x) = 4x^3 - 27x^2 + 50x - 27$ 1
$P'(1) = 4 - 27 + 50 - 27$
$= 0$
$x = 1$ is also a zero of $P'(x)$.
$\therefore x = 1$ is a double zero of $P(x)$.

(ii) Hence, or otherwise, factorise $P(x)$ into linear factors. **2**

$x = 1$ is a double zero.
$\therefore (x-1)^2 = x^2 - 2x + 1$ is a factor of $P(x)$.

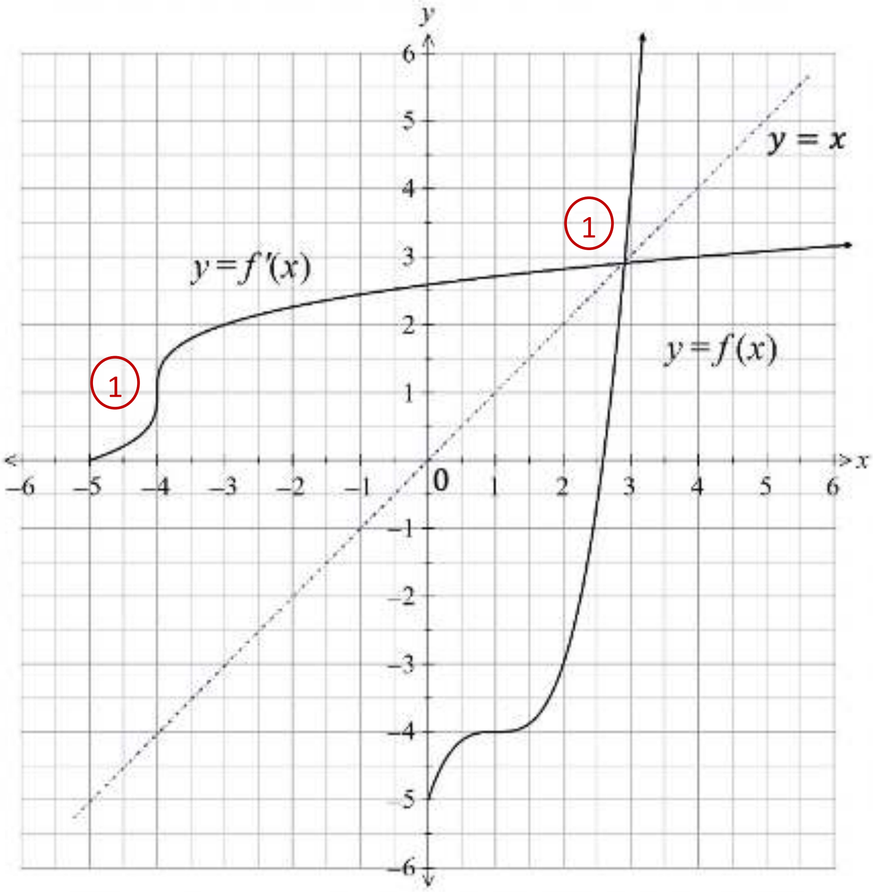
$$\begin{array}{r l}
 & x^2 - 7x + 10 \\ \cline{2-2}
x^2 - 2x + 1 \,\big) & x^4 - 9x^3 + 25x^2 - 27x + 10 \\
 & \underline{x^4 - 2x^3 + \quad x^2} \\
 & -7x^3 + 24x^2 - 27x \\
 & \underline{-7x^3 + 14x^2 - 7x} \\
 & 10x^2 - 20x + 10 \\
 & \underline{10x^2 - 20x + 10} \\
 & 0
\end{array}$$
1

$P(x) = (x-1)^2(x^2 - 7x + 10)$
$= (x-1)^2(x-2)(x-5)$ 1

Question 12 (15 marks)
Use the **Question 12 Writing Booklet**

(a) (i) The graph of $f(x) = (x-1)^3 - 4$ for $x \geq 0$ is shown. **2**

On the same diagram, sketch the graph of $y = f^{-1}(x)$.



(ii) Write the equation that describes $f^{-1}(x)$ and state its domain. **2**

$$f(x): \quad y = (x-1)^3 - 4\,,\ x \geq 0$$

Swap x and y to obtain $f^{-1}(x)$: $x = (y-1)^3 - 4\,,\ y \geq 0$

$$x + 4 = (y-1)^3$$

$$\sqrt[3]{x+4} = y - 1$$

$$f^{-1}(x) = \sqrt[3]{x+4} + 1 \qquad 1$$

The domain of $f^{-1}(x)$ is the range of $f(x)$.

Hence, the domain of $f^{-1}(x)$ is $x \geq -5$. 1

(b) (i) Use the t-formulae to show that **2**

$$\sin x - \tan\frac{x}{2} = \tan\frac{x}{2}\cos x.$$

Let $t = \tan\frac{x}{2}$. Then $\sin x = \frac{2t}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$.

$$\begin{aligned} LHS &= \sin x - \tan\frac{x}{2} \\ &= \frac{2t}{1+t^2} - t \quad (1) \\ &= \frac{2t - t(1+t^2)}{1+t^2} \\ &= \frac{t(2-1-t^2)}{1+t^2} \\ &= \frac{t(1-t^2)}{1+t^2} \quad (1) \\ &= \tan\frac{x}{2}\cos x \\ &= RHS \end{aligned}$$

(ii) Hence, or otherwise, solve the equation **2**

$$\sin x - \tan\frac{x}{2} = 0 \ \text{ for } \ 0 \le x \le 2\pi.$$

$\sin x - \tan\frac{x}{2} = \tan\frac{x}{2}\cos x$ (from part i) Note: $x \neq \pi$.

$\tan\frac{x}{2}\cos x = 0, \quad 0 \le x \le 2\pi$

$\tan\frac{x}{2} = 0$ for $0 \le \frac{x}{2} \le \pi$ or $\cos x = 0$ for $0 \le x \le 2\pi$

$\frac{x}{2} = 0, \pi$ or $x = \frac{\pi}{2}, \frac{3\pi}{2}$ (1)

$x = 0, 2\pi$ (1)

$\therefore x = 0, \frac{\pi}{2}, \frac{3\pi}{2}$ or 2π

(c) Use the method of separation of variables to solve the differential equation $\frac{dy}{dx} = \frac{3y}{x}$, given that $\frac{dy}{dx} = 6$ when $x = 2$. **3**

$$\frac{dy}{dx} = \frac{3y}{x}$$

$$\frac{1}{y}dy = \frac{3}{x}dx$$

$$\int \frac{1}{y}dy = 3\int \frac{1}{x}dx \quad (1)$$

$$\ln|y| = 3\ln|x| + C$$

$$\ln|y| = \ln|x^3| + C$$

$$y = \pm e^{\ln x^3 + C}$$

$y = Ax^3$ where $A = \pm e^C$ (1)

When $x = 2, y = 4$.

$$4 = A(2^3)$$

$$A = \frac{1}{2}$$

$$y = \frac{1}{2}x^3 \quad (1)$$

When $x = 2, \frac{dy}{dx} = 6$

$$6 = \frac{3y}{2}$$

$$y = 4$$

Alternatively, since $x = 2$ and $y = 4$ are positive,

$\ln 4 = 3\ln 2 + C.$

$C = \ln 4 - \ln 8$

$C = \ln\frac{1}{2}$

$\ln y = \ln x^3 + \ln\frac{1}{2}$

$\ln y = \ln\frac{1}{2}x^3$

$y = \frac{1}{2}x^3$

(d) Ten people including Bob and Mary are seated randomly around a circular table. What is the probability that Bob and Mary do not sit next to each other? **2**

The total number of arrangements with no restrictions: 9!

Consider the scenario where Bob and Mary sit together, as 1 entity. Plus 8 others, which makes 9 entities, arranged in a circle: 8! ways

Bob and Mary may swap seats: 2! ways 1

$$\begin{aligned}\text{P(Bob and Mary NOT together)} &= 1 - \text{P(Bob and Mary together)}\\ &= 1 - \frac{8!2!}{9!}\\ &= 1 - \frac{2}{9}\\ &= \frac{7}{9}\end{aligned}$$ 1

(e) The polynomial equation $x^3 - 5x^2 + 2x - 12 = 0$ has roots α, β and γ. **2**

Find the equation of the polynomial with roots $2\alpha,\ 2\beta$ and 2γ.

<u>Method 1:</u>

Let $x = 2\alpha$ be a root of the new polynomial equation.

$\alpha = \frac{x}{2}$, which is a root of $x^3 - 5x^2 + 2x - 12 = 0$.

$$\therefore \left(\frac{x}{2}\right)^3 - 5\left(\frac{x}{2}\right)^2 + 2\left(\frac{x}{2}\right) - 12 = 0$$

$$\frac{x^3}{8} - \frac{5x^2}{4} + x - 12 = 0$$

$$x^3 - 10x^2 + 8x - 96 = 0$$

<u>Method 2:</u>

α, β and γ are the roots of $x^3 - 5x^2 + 2x - 12 = 0$.

$$\alpha + \beta + \gamma = -\frac{b}{a} = 5$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} = 2$$ 1

$$\alpha\beta\gamma = -\frac{d}{a} = 12$$

The equation with roots $2\alpha,\ 2\beta$ and 2γ is:

$$\begin{aligned}x^3 - (2\alpha + 2\beta + 2\gamma)x^2 + (2\alpha 2\beta + 2\alpha 2\gamma + 2\beta 2\gamma)x - (2\alpha 2\beta 2\gamma) &= 0\\ x^3 - 2(\alpha + \beta + \gamma)x^2 + 4(\alpha\beta + \alpha\gamma + \beta\gamma)x + 8\alpha\beta\gamma &= 0\\ x^3 - 2(5)x^2 + 4(2)x - 8(12) &= 0\\ x^3 - 10x^2 + 8x - 96 &= 0\end{aligned}$$ 1

Question 13 (15 marks)
Use a new writing booklet

(a) (i) Prove **1**

$$\sin x + \sin 2x + \sin 3x = \sin 2x\,(2\cos x + 1).$$

$$\begin{aligned} LHS &= \sin(2x+x) + \sin(2x-x) + \sin(2x) \\ &= 2\sin 2x\cos x + \sin 2x \quad \text{1} \\ &= \sin 2x(2\cos x + 1) \\ &= RHS \end{aligned}$$

(ii) Hence, solve $\sin x + \sin 2x + \sin 3x = 0$ for $x \in \left[0, \frac{\pi}{2}\right]$. **2**

$\sin x + \sin 2x + \sin 3x = 0$

$\sin 2x(2\cos x + 1) = 0$

$\sin 2x = 0$ for $0 \le x \le \pi$, or $2\cos x + 1 = 0$ for $0 \le x \le \frac{\pi}{2}$

$2x = 0, \pi$ $\qquad$ $\cos x = -\frac{1}{2}$

$x = 0, \frac{\pi}{2}$ 1 $\qquad$ No solution in the given domain. 1

$\therefore x = 0$ or $\frac{\pi}{2}$

(b) Find, in simplest exact form, the coordinates of the stationary point(s) on the curve **3**

$$y = x^2 + \cos^{-1} x.$$

$\frac{dy}{dx} = 2x - \frac{1}{\sqrt{1-x^2}}$, $-1 < x < 1$

$= \frac{2x\sqrt{1-x^2}-1}{\sqrt{1-x^2}} = 0$ for stationary point

$2x\sqrt{1-x^2} - 1 = 0$

$2x\sqrt{1-x^2} = 1$, $x > 0$ since $\sqrt{1-x^2} > 0$

$4x^2(1-x^2) = 1$ 1

$4x^2 - 4x^4 = 1$

$0 = 4x^4 - 4x^2 + 1$

$0 = (2x^2-1)^2$

$2x^2 = 1$

$x = \pm\frac{1}{\sqrt{2}}$

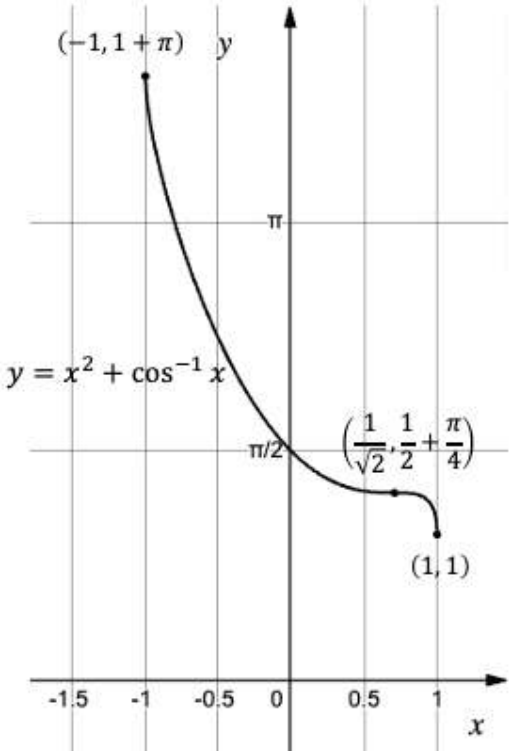
But $x > 0$. So $x = \frac{1}{\sqrt{2}}$. 1

$y = \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^{-1}\frac{1}{\sqrt{2}}$

$y = \frac{1}{2} + \frac{\pi}{4}$ 1

There is one stationary point at $\left(\frac{1}{\sqrt{2}}, \frac{1}{2} + \frac{\pi}{4}\right)$.

The graph of $y = x^2 + \cos^{-1} x$ is provided for illustration purposes (not mandated).



(c) (i) Prove the combinatorial identity: **1**

$$^{n+1}C_{r+1} = {}^nC_r + {}^nC_{r+1}$$

$$\begin{aligned}
RHS &= {}^nC_r + {}^nC_{r+1} \\
&= \frac{n!}{r!(n-r)!} + \frac{n!}{(r+1)![n-(r+1)]!} \\
&= \frac{(r+1)n!}{(r+1)!(n-r)!} + \frac{(n-r)n!}{(r+1)!(n-r)(n-r-1)!} \\
&= \frac{(r+1+n-r)\,n!}{(r+1)!(n-r)!} \\
&= \frac{(n+1)!}{(r+1)![n+1-(r+1)]!} \qquad 1 \\
&= {}^{n+1}C_{r+1} \\
&= LHS
\end{aligned}$$

(ii) Given $p = {}^nC_4$ and $q = {}^nC_5$, show that: **2**

$$^nC_{n-4} + {}^{n+1}C_{n-5} = p + \frac{q(n+1)}{6}$$

Method 1:

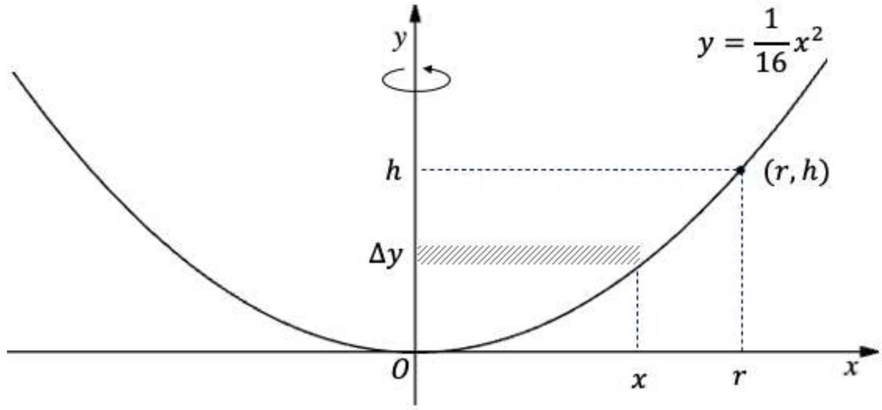
$$\begin{aligned}
LHS &= {}^nC_{n-4} + {}^{n+1}C_{n-5} \\
&= {}^nC_4 + \frac{(n+1)!}{(n-5)![n+1-(n-5)]!} \\
&= p + \frac{(n+1)!}{(n-5)!\,6!} \qquad 1 \\
&= p + \frac{n!}{5!(n-5)!} \times \frac{n+1}{6} \\
&= p + {}^nC_5 \times \frac{n+1}{6} \qquad 1 \\
&= p + q\left(\frac{n+1}{6}\right) \\
&= RHS
\end{aligned}$$

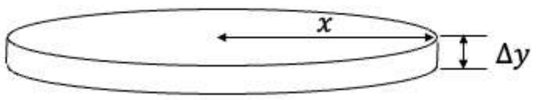
Method 2:

$$\begin{aligned}
RHS &= p + \frac{q(n+1)}{6} \\
&= \binom{n}{4} + \binom{n}{5} \times \frac{n+1}{6} \\
&= \binom{n}{n-4} + \frac{n!\,(n+1)}{5!(n-5)! \times 6} \\
&= \binom{n}{n-4} + \frac{(n+1)!}{6!(n+1-6)!} \\
&= \binom{n}{n-4} + \binom{n+1}{n+1-6} \\
&= \binom{n}{n-4} + \binom{n+1}{n-5} \\
&= LHS
\end{aligned}$$

(d) The shape of a water reservoir is created by rotating the parabola $y = \frac{1}{16}x^2$ about the y-axis between y = 0 and y = h.

(i) Show that the volume of the water is given by $V = \frac{A^2}{32\pi}$. **2**





$\Delta V = \pi x^2 \Delta y$

$$y = \frac{1}{16}x^2$$
$$16y = x^2$$
$$\Delta V = \pi x^2 \Delta y$$
$$\Delta V = \pi 16y \Delta y$$
$$V = 16\pi \int_0^h y\, dy$$
$$= 8\pi[y^2]_0^h$$
$$= 8\pi(h^2 - 0)$$
$$= 8\pi h^2 \quad 1$$

Also, $A = \pi r^2$ and $r^2 = 16h$.

$$\therefore A = \pi(16h)$$
$$h = \frac{A}{16\pi}$$

Now, $V = 8\pi h^2$

$$= 8\pi\left(\frac{A}{16\pi}\right)^2$$
$$= \frac{8\pi A^2}{(2\times 8\pi)^2} \quad 1$$
$$V = \frac{A^2}{32\pi}$$

(ii) The area A is decreasing at a constant rate of $\frac{\pi}{20} m^2/h$. **2**

Show that the differential equation below represents the rate at which the water volume is decreasing:

$$\frac{dV}{dt} = -\frac{1}{80}\sqrt{2\pi V}$$

$V = \frac{A^2}{32\pi} \quad \Rightarrow \quad \frac{dV}{dA} = \frac{A}{16\pi}$

$\frac{dV}{dt} = \frac{dV}{dA} \times \frac{dA}{dt}$ where $\frac{dA}{dt} = -\frac{\pi}{20}$ (given)

$= \frac{A}{16\pi}\left(-\frac{\pi}{20}\right)$ 1

$= -\frac{A}{320}$, where $A^2 = 32\pi V$

$= -\frac{1}{320}\sqrt{32\pi V},\ A \geq 0$ 1

$\frac{dV}{dt} = -\frac{1}{80}\sqrt{2\pi V}$

(iii) Initially, the volume of the water is $20\,000\ m^3$. **2**

By solving the differential equation, find how long it takes for the water volume to fall below $10\,000\ m^3$. Express your answer correct to the nearest hour.

$$\frac{dV}{dt} = -\frac{1}{80}\sqrt{2\pi V}$$

$$\frac{dV}{\sqrt{V}} = -\frac{\sqrt{2\pi}}{80}dt$$

Method 1:

$$\int_{20\,000}^{10\,000} V^{-\frac{1}{2}}\,dV = -\frac{\sqrt{2\pi}}{80}\int_0^T dt \quad 1$$

$$2\left[\sqrt{V}\right]_{20\,000}^{10\,000} = -\frac{\sqrt{2\pi}}{80}[t]_0^T$$

$$2\left(\sqrt{10\,000} - \sqrt{20\,000}\right) = -\frac{\sqrt{2\pi}}{80}T$$

$$\frac{160(100\sqrt{2} - 100)}{\sqrt{2\pi}} = T$$

$$T = 2643.956\,850\ldots$$

$$T \approx 2644 \text{ hours} \quad 1$$

It takes 2644 hours for the volume of water to fall below 10 000 L.

Method 2:

$$\int V^{-\frac{1}{2}}\,dV = -\frac{\sqrt{2\pi}}{80}\int dt$$

$$2\sqrt{V} = -\frac{\sqrt{2\pi}}{80}t + C$$

When $t = 0, V = 20\,000$.

$$2\sqrt{20\,000} = -\frac{\sqrt{2\pi}}{80} \times 0 + C$$

$$C = 200\sqrt{2}$$

$$2\sqrt{V} = -\frac{\sqrt{2\pi}}{80}t + 200\sqrt{2} \quad 1$$

$$\sqrt{V} = -\frac{\sqrt{2\pi}}{160}t + 100\sqrt{2}$$

When $V < 10\,000$

$$\sqrt{V} < 100$$

$$-\frac{\sqrt{2\pi}}{160}t + 100\sqrt{2} < 100$$

$$100(\sqrt{2} - 1) < \frac{\sqrt{2\pi}}{160}t$$

$$\frac{16\,000(\sqrt{2}-1)}{\sqrt{2\pi}} < t$$

$$t > 2643.956\,850\ldots$$

$$\therefore t = 2644 \text{ hours} \quad 1$$

Question 14 (15 marks)
Use a new writing booklet

(a) Find the value of $\int_1^{49} \frac{1}{\sqrt{x}\sqrt{1+\sqrt{x}}} dx$ using the substitution $u = 1 + \sqrt{x}$. **4**

Express your answer in the form $a\sqrt{b}$ where a and b are both positive integers.

Let $u = 1 + \sqrt{x}$

$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$

$2du = \frac{dx}{\sqrt{x}}$ 1

When $x = 1,\ u = 2.$

When $x = 49, u = 8.$

$$\int_1^{49} \frac{1}{\sqrt{x}\sqrt{1+\sqrt{x}}} dx = 2\int_2^8 \frac{du}{\sqrt{u}}$$ 1

$$= 2\int_2^8 u^{-\frac{1}{2}} du$$

$$= 2 \times 2\left[\sqrt{u}\right]_2^8$$ 1

$$= 4(2\sqrt{2} - \sqrt{2})$$

$$= 4\sqrt{2}$$ 1

(b) Use Mathematical Induction to prove, for all integers $n \geq 1$, that **3**

$$\left(1 \times 1! + \frac{0}{1!}\right) + \left(2 \times 2! + \frac{1}{2!}\right) + \left(3 \times 3! + \frac{2}{3!}\right) + \cdots + \left(n \times n! + \frac{n-1}{n!}\right) = (n+1)! - \frac{1}{n!}.$$

Proof:

When $n = 1$, $LHS = 1 \times 1! + \frac{0}{1!} = 1$ $\quad RHS = (1+1)! - \frac{1}{1!} = 1$ 1

$\therefore$ The statement is true for $n = 1$.

Assume the statement is true for $n = k,\ k \in \mathbb{Z}^+$

$$\left(1 \times 1! + \frac{0}{1!}\right) + \left(2 \times 2! + \frac{1}{2!}\right) + \left(3 \times 3! + \frac{2}{3!}\right) + \cdots + \left(k \times k! + \frac{k-1}{k!}\right) = (k+1)! - \frac{1}{k!}$$

Show true for $n = k + 1$.

RTP: $\left(1 \times 1! + \frac{0}{1!}\right) + \left(2 \times 2! + \frac{1}{2!}\right) + \left(3 \times 3! + \frac{2}{3!}\right) + \cdots + \left(k \times k! + \frac{k-1}{k!}\right)$

$$+ \left[(k+1)(k+1)! + \frac{(k+1)-1}{(k+1)!}\right] = (k+2)! - \frac{1}{(k+1)!}$$

$LHS = (k+1)! - \frac{1}{k!} + (k+1)(k+1)! + \frac{k}{(k+1)!}$ by assumption 1

$$= (k+1)! + (k+1)(k+1)! + \frac{k}{(k+1)!} - \frac{k+1}{(k+1)k!}$$

$$= (k+1)!\,(1 + k + 1) + \frac{k-(k+1)}{(k+1)!}$$

$$= (k+2)(k+1)! + \frac{-1}{(k+1)!}$$ 1

$$= (k+2)! - \frac{1}{(k+1)!}$$

$$= RHS$$

So, if the statement is true for $n = k$, then it is true for $n = k + 1$.

By mathematical induction, the statement is true for all integers $n \geq 1$.

(c) A particle is projected from a point O on the horizontal plane with velocity V and at an angle θ to the horizontal.

(i) Find H, the greatest height reached in terms of V, θ and g, where g is the acceleration due to gravity. Show necessary working. **2**

$\frac{d^2y}{dt^2} = -g$

$\frac{dy}{dt} = -gt + V\sin\theta \quad (\frac{dy}{dt} = V\sin\theta \text{ when } t = 0)$

$y = -\frac{1}{2}gt^2 + Vt\sin\theta \quad (y = 0 \text{ when } t = 0)$

$y = H$ when $\frac{dy}{dt} = 0$

So, $-gt + V\sin\theta = 0$

$V\sin\theta = gt$

$t = \frac{V\sin\theta}{g}$ 1

$H = -\frac{1}{2}g\left(\frac{V\sin\theta}{g}\right)^2 + V\left(\frac{V\sin\theta}{g}\right)\sin\theta$

$H = \frac{V^2\sin^2\theta}{2g}$ 1

$\therefore V\sin\theta = \sqrt{2gH}$

(ii) A point A on the trajectory is at height h above the plane and is such that the particle passes through A before reaching its highest point. Show that the time taken to travel from O to A is **2**

$$\sqrt{\frac{2H}{g}} - \sqrt{\frac{2(H-h)}{g}}.$$

$$h = -\frac{1}{2}gt^2 + Vt\sin\theta$$

$gt^2 - (2V\sin\theta)t + 2h = 0$

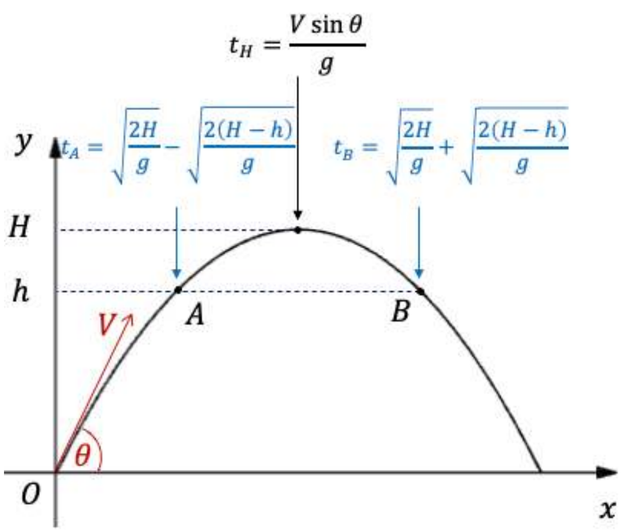
$$t = \frac{V\sin\theta \pm \sqrt{V^2\sin^2\theta - 2gh}}{g}$$ 1

$t = \frac{\sqrt{2gH} \pm \sqrt{2gH - 2gh}}{g}$

Time taken from O to A:

$t_A = \frac{\sqrt{2gH} - \sqrt{2g(H-h)}}{g}$ 1

$t_A = \sqrt{\frac{2H}{g}} - \sqrt{\frac{2(H-h)}{g}}$



(iii) B is the other point on the trajectory which is at the height h above the plane. If the time taken to travel from A to B is equal to the time taken to travel from O to the highest point, show that $h = \frac{3}{4}H$. **2**

$$t_{AB} = \sqrt{\frac{2H}{g}} + \sqrt{\frac{2(H-h)}{g}} - \left(\sqrt{\frac{2H}{g}} - \sqrt{\frac{2(H-h)}{g}}\right)$$

$$= 2\left(\sqrt{\frac{2(H-h)}{g}}\right) = \sqrt{\frac{8(H-h)}{g}} \qquad 1$$

Time taken from O to the highest point, H, is

$$t_H = \frac{V\sin\theta}{g}$$

$$= \frac{\sqrt{2gH}}{g} = \sqrt{\frac{2H}{g}}$$

$$\because t_{AB} = t_H$$

$$\sqrt{\frac{8(H-h)}{g}} = \sqrt{\frac{2H}{g}} \qquad 1$$

$$8(H-h) = 2H$$

$$4H - 4h = H$$

$$3H = 4h$$

$$h = \frac{3}{4}H$$

(iv) Hence, show that the gradient of the trajectory at A is given by **2**

$$\frac{dy}{dx} = \frac{1}{2}tan\,\theta\,.$$

<u>Method 1:</u>

At the point A,

$t = \sqrt{\frac{2H}{g}} - \sqrt{\frac{2(H-h)}{g}}$,

$h = \frac{3}{4}H$ and

$V^2\sin^2\theta = 2gH$

or $V\sin\theta = \sqrt{2gH}$

$$\frac{dy}{dt} = -gt + V\sin\theta$$

$$= -g\left(\sqrt{\frac{2H}{g}} - \sqrt{\frac{2(H-h)}{g}}\right) + V\sin\theta$$

$$= \sqrt{2g(H-h)} - \sqrt{2gH} + \sqrt{2gH}$$

$$= \sqrt{2g\left(H - \frac{3H}{4}\right)}$$

$$= \sqrt{2g\left(\frac{H}{4}\right)}$$

$$= \frac{1}{2}\sqrt{2gH} \qquad 1$$

$$= \frac{1}{2}V\sin\theta$$

$$\frac{dx}{dt} = V\cos\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{\frac{1}{2}V\sin\theta}{V\cos\theta} \qquad 1$$

$$= \frac{1}{2}\tan\theta$$

Method 2:

$y = -\frac{1}{2}gt^2 + (V\sin\theta)\,t \quad ...(*)$

$x = (V\cos\theta)t$

$t = \frac{x}{V\cos\theta}$. Substitute this into $(*)$:

$y = -\frac{1}{2}g\left(\frac{x}{V\cos\theta}\right)^2 + (V\sin\theta)\left(\frac{x}{V\cos\theta}\right)$

$y = -\frac{g}{2V^2\cos^2\theta}x^2 + (\tan\theta)x$

$\frac{dy}{dx} = -\frac{g}{V^2\cos^2\theta}x + \tan\theta$

$\frac{dy}{dx} = -\frac{g}{V^2\cos^2\theta}(V\cos\theta)\,t + \tan\theta\,,\quad$ where $x = (V\cos\theta)\,t$

$\frac{dy}{dx} = -\frac{g}{V\cos\theta}t + \tan\theta \quad ...\ (*)$ 1

At A, $t = \dfrac{V\sin\theta - \sqrt{V^2\sin^2\theta - 2gh}}{g}$ from part (ii)

Substitute t into $(*)$:

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{g}{V\cos\theta}\times\frac{V\sin\theta-\sqrt{V^2\sin^2\theta-2gh}}{g} + \tan\theta \\
&= \frac{-V\sin\theta+\sqrt{V^2\sin^2\theta-2gh}}{V\cos\theta} + \tan\theta \\
&= -\tan\theta + \sqrt{\frac{V^2\sin^2\theta-2gh}{V^2\cos^2\theta}} + \tan\theta \\
&= \sqrt{\tan^2\theta - \frac{2gh}{V^2\cos^2\theta}} \\
&= \sqrt{\tan^2\theta - \frac{h}{V^2\cos^2\theta}\times 2g}
\end{aligned}$$

$V^2\sin^2\theta = 2gH$

$\therefore 2g = \frac{V^2\sin^2\theta}{H}$

$$\begin{aligned}
&= \sqrt{\tan^2\theta - \frac{h}{V^2\cos^2\theta}\times\frac{V^2\sin^2\theta}{H}} \\
&= \sqrt{\tan^2\theta - \frac{\mathrm{h}}{\mathrm{H}}\tan^2\theta}
\end{aligned}$$ 1

$= \tan\theta\sqrt{1-\frac{h}{H}}$, where $\frac{h}{H} = \frac{3}{4}$ from part (iii)

$= \tan\theta\sqrt{1-\frac{3}{4}}$

$= \frac{1}{2}\tan\theta$

End of paper

2024 Trial HSC Examination
Mathematics Extension 1 Course

NESA Number: ANSWERS Teacher ____________________

Section I – Multiple Choice Answer Sheet

Allow about 15 minutes for this section

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample:	2 + 4 =	(A) 2	(B) 6	(C) 8	(D) 9
		A ○	B ●	C ○	D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

		A ●	B ●	C ○	D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word **correct** and drawing an arrow as follows.

		A ●	correct B ●	C ○	D ○

1.	A ○	B ●	C ○	D ○
2.	A ●	B ○	C ○	D ○
3.	A ○	B ●	C ○	D ○
4.	A ○	B ○	C ○	D ●
5.	A ○	B ○	C ●	D ○
6.	A ○	B ●	C ○	D ○
7.	A ○	B ○	C ○	D ●
8.	A ●	B ○	C ○	D ○
9.	A ○	B ○	C ●	D ○
10.	A ●	B ○	C ○	D ○