Name : ______________________

Class : 12 MTX___

CHERRYBROOK TECHNOLOGY HIGH SCHOOL

2002 AP4

YEAR 12 TRIAL HSC

MATHEMATICS EXTENSION I

Time allowed - 2 hours
(plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES:

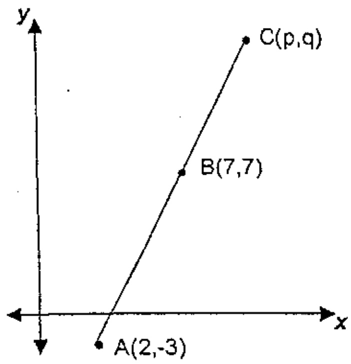
- Attempt ALL questions.
- All questions are of equal value
- Standard Integrals are provided.
- Approved calculators may be used.
- All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.
- Each question attempted is to be returned on a new page clearly marked Question 1, Question 2, etc on the top of the page.
- Each page must show your class and your name.

Students are advised that this is a school based Trial Examination *only* and cannot in any way guarantee the complete content nor format of the Higher School Certificate Examination.

QUESTION ONE. MARKS

(a) Differentiate $3x\cos^{-1}x$. 2

(b) Find the co-ordinates of the point $C(p,q)$ below, given that $AC:CB = 8:3$. 2



(c) Find the size of the acute angle between the tangents of $y = \tan^{-1}x$ at the points where $x = 0$ and $x = \frac{1}{2}$. 3

(d) Use the substitution $u = x^2 - 1$ to evaluate $\int_0^1 3x(x^2-1)^5\,dx$. 3

(e) Find the value of k if $x+4$ is a factor of $P(x) = 2x^3 + 3x^2 + kx - 12$. 2

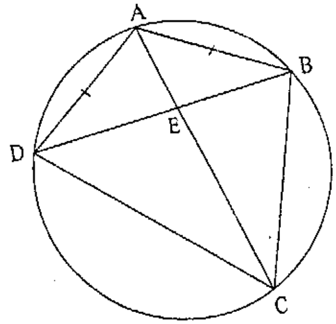
QUESTION TWO. START A NEW PAGE.

(a) Solve $\dfrac{x+3}{x-1} \le 2$ 3

(b) In how many ways can the letters in the word **COMMONWEALTH** be arranged if the **C** still occupies the first position and the **H** still occupies the last position? 2

QUESTION TWO continued. MARKS

(c)



ABCD is a cyclic quadrilateral with AB = AD. The diagonals AC and BD intersect at E.

(i) Show that Δ BEC is similar to Δ ADC . 3

(ii) Show that BE $\times$ AC = AB $\times$ BC. 1

(d) Ten people arrive to eat at a restaurant. The only seating available for them is at two circular tables, one that seats six persons and another that seats four.

(i) Using these tables, show that there are 151200 different seating arrangements for the ten people. 1

(ii) Assuming that the seating arrangement is random, what is the probability that a particular couple will be seated at the same table? 2

QUESTION THREE. START A NEW PAGE.

(a) Prove by induction that $\frac{2}{3}+\frac{10}{3}+8+\ldots\ldots\ldots\ldots\frac{n}{3}(3n-1)=\frac{1}{3}n^2(n+1)$ 3

for $n =$ 1, 2, 3................

(b) Consider the function $y=2\sin^{-1}\frac{x}{4}$.

(i) State the domain. 1

(ii) Sketch the graph of the function. 2

(iii) Find the gradient of the curve at the point where $y=\frac{\pi}{3}$. 2

MARKS

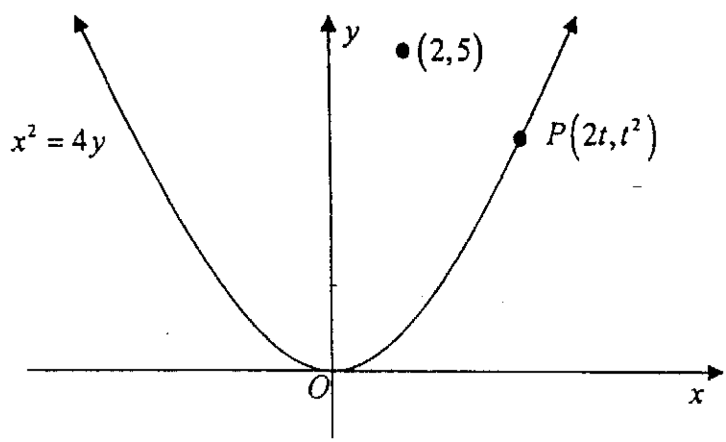
QUESTION THREE continued.

(c) (i) Express $\sqrt{3}\cos x+\sin x$ in the form $k\cos(x-\alpha)$, where $k>0$ and $0<x<\dfrac{\pi}{2}$. 2

(ii) Hence solve the equation $\sqrt{3}\cos x+\sin x=1$ for $0\le x\le 2\pi$. 2

QUESTION FOUR. **START A NEW PAGE.**

(a)



The above diagram is of the parabola $x^2=4y$. $P(2t,t^2)$ is a variable point on the parabola.

(i) Show that the normal at P has gradient $-\dfrac{1}{t}$. 2

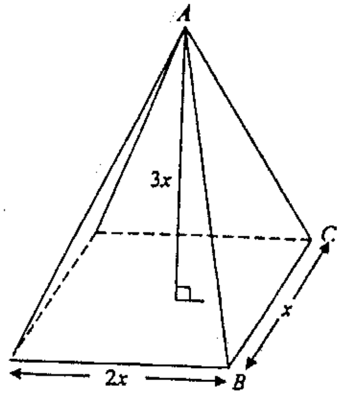
(ii) Show that the normal at P has equation $x+ty=t^3+2t$. 1

(iii) The normal at P passes through the fixed point $(2,5)$. Show that $t^3-3t-2=0$. 1

(iv) Hence, find the two points on the parabola at which the normals pass through the point $(2,5)$. 4

QUESTION FOUR continued. MARKS

(b)



The diagram shows a rectangular pyramid whose base is $2x$ units long and x units wide, and whose perpendicular height is $3x$ units.

Find, correct to the nearest minute:

(i) the angle between a slant edge and the rectangular base. 3

(ii) the angle between the side face ABC and the rectangular base. 1

QUESTION FIVE. START A NEW PAGE.

(a) Consider the function $g(x) = \dfrac{1}{1+e^x}$.

(i) Show that $g'(x) < 0$ for all x. 1

(ii) Sketch the graph of $y = g(x)$. 2

(iii) Find the inverse function $y = g^{-1}(x)$. 2

(b) The velocity V in m/s of a particle moving in a straight line is given by $V = 2 - \dfrac{1}{4t+1}$, where t is the time in seconds.

(i) Find the initial velocity of the particle. 1

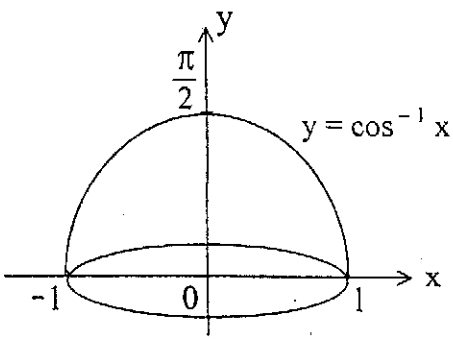
(ii) Does the particle have a limiting velocity? Give reasons for your answer. 2

(iii) Find an expression for the acceleration of the particle in terms of t. 2

(iv) If it is known that the particle starts from the origin, does it ever return to the origin? Give reasons for your answer. 2

QUESTION SIX. START A NEW PAGE.

(a) A solid is formed, by rotating about the y-axis, the region bounded by the curve $y = \cos^{-1} x$, the x-axis and the y-axis, as shown in the diagram.



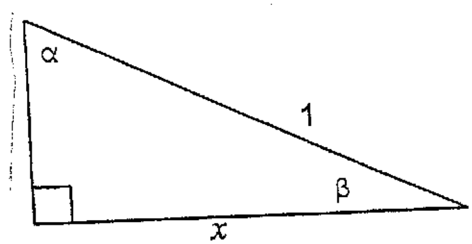
(i) Show that the volume of the solid is given by 1

$$V = \pi \int_0^{\frac{\pi}{2}} \cos^2 y \; dy$$

(ii) Calculate the volume of the solid. 3

(b) The function $g(x) = 20\log_e x - x^2$ has a zero near $x = 5$. Use Newton's method with one approximation to the zero. Express your answer correct to 4 significant figures. 3

(c) (i) Using the right triangle shown below, or otherwise, show that $\sin^{-1} + \cos^{-1} = \dfrac{\pi}{2}$ 2



(ii) Hence evaluate $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{3}} \sin^{-1} x + \cos^{-1} x \; dx$. 1

QUESTION SIX continued. MARKS

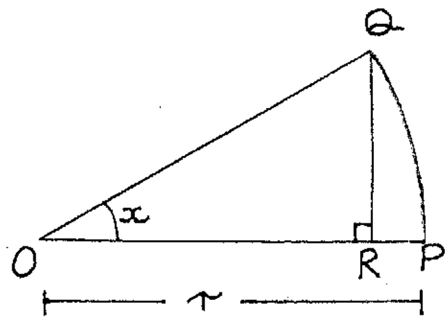
(c) The polynomial $P(x) = x^4 + x^3 + x^2 + x - 2$ has roots α, β, χ and δ. If $\alpha = 1$, find the value of 2

$\frac{1}{\beta} + \frac{1}{\chi} + \frac{1}{\delta}$.

QUESTION SEVEN. START A NEW PAGE.

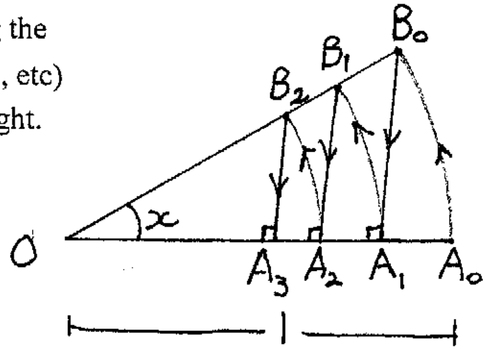
(a) The acceleration, a ms^{-1}, of a particle moving in a straight line is given by the equation, $a = \frac{x^3}{8} + \frac{x}{8}$, where x is the displacement in metres of the particle from the origin. The velocity of the particle at any time t, is given by v. 3

(b) (i) PQ is the arc of a circle with radius r subtending an acute angle x, in radians, at the centre 0. R is the foot of the perpendicular from Q to the radius OP. Find the length of the arc PQ and the interval QR in terms of x and r. 2



(ii) An ant travels from A_0 to 0 along the *saw tooth path* (A_0 to B_0, B_0 to A_1, etc) as shown in the diagram on the right. Show that the total distance 'y' travelled by the ant is 4

$$y = \frac{x + \sin x}{1 - \cos x}.$$



(iii) Given $0 < x \leq \frac{\pi}{2}$, use the derivative of y to find the value of x that gives the shortest such distance. 3

END OF TEST.

(a) $\dfrac{d\,3x\cos^{-1}x}{dx} = vu' + vu'$ where $u = 3x$

$u' = 3$

$v = \cos^{-1}x$

$v' = \dfrac{-1}{\sqrt{1-x^2}}$

① ①

$= \cos^{-1}x \cdot 3 + \dfrac{-1}{\sqrt{1-x^2}}\,3$

$= 3\cos^{-1}x - \dfrac{3}{\sqrt{1-x^2}}$

(b) $C(p,q)$ is an external pt to AB so use ratio $8:-3$

$p = \dfrac{8(7)-3(2)}{8-3}$ $\quad q = \dfrac{8(7)-3(-3)}{8-3}$

$= \dfrac{56-6}{5}$ $\quad = \dfrac{56+9}{5}$

$= 10$ ① $\quad = 13$ ①

$\therefore C(p,q)$ is $(10, 13)$

(c) $y = \tan^{-1}x$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{1+x^2}$

At $x = 0$, $m = 1$

At $x = \frac{1}{2}$, $m = \dfrac{1}{1+\frac{1}{4}} = \dfrac{4}{5}$ ①

$\therefore \tan\alpha = \left|\dfrac{1-\frac{4}{5}}{\ldots}\right| = \dfrac{1}{9}$ ①

$\therefore \alpha = 6°\,20'$ to nearest minute ①

(d) $u = x^2 - 1$

So $\dfrac{du}{dx} = 2x$

ie $du = 2x\,dx$.

If $x = 1$ $u = 0$ ①

$x = 0$ $u = -1$

So $\displaystyle\int_0^1 3x(x^2-1)^5\,dx = \int_{-1}^{0} u^5 \frac{3}{2}\,du$ ①

$= \dfrac{3}{2}\displaystyle\int_{-1}^{0} u^5\,du$

$= \dfrac{3}{2}\left[u^6/6\right]_{-1}^{0}$

$= -\dfrac{1}{4}$ ①

(e) If $(x+4)$ is a factor, then

$x = -4$ gives $P(x) = 0$

$P(-4) = 2(-4)^3 + 3(-4)^2 - 3k - 12 = 0$ ①

$0 = -128 + 48 - 4k - 12$

$4k = -92$

$k = -23$ ①

TOTAL: 12.

STION 2 CONTINUED.

) No. of arrangements with couple at large table is $^8C_4 \times 5! \times 3!$

$= 50400$

No. of arrangements with couple at small table is $^8C_2 \times 5! \times 3!$

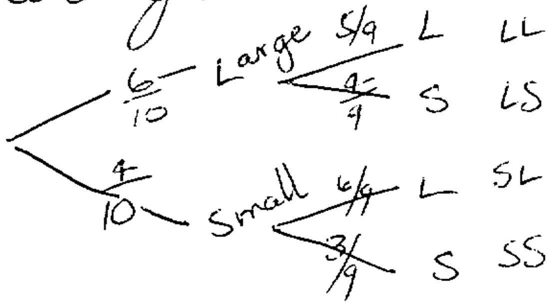
$= 20160$. (1)

P(couple at same table)

$= \dfrac{50400 + 20160}{151200}$

$= 7/15$ (1)

OR tree diagram



$P(LL \text{ or } SS) = \frac{6}{10} \times \frac{5}{9} + \frac{4}{10} \times \frac{3}{9}$

$= 7/15$.

TOTAL 12 MARKS

QUESTION 3:

(a) Test for $n = 1$,

$RHS = \frac{1}{3}(1)(2)$ LHS. 1st term $= \frac{2}{3}$ (1)

$= \frac{2}{3}$

$\therefore$ LHS = RHS $n=1$ true

Let $n = k$. Assume true for $n = k$.

So $S_k = \frac{2}{3} + \frac{10}{3} + 8 \ldots \frac{k}{3}(3k-1) = \frac{1}{3}k^2(k+1)$

Now when $n = k+1$ we have (1)

$S_{k+1} = \frac{2}{3} + \frac{10}{3} + 8 + \ldots + \frac{k}{3}(3k-1) + \frac{k+1}{3}(3(k+1)$

$= \frac{1}{3}k^2(k+1) + \frac{k+1}{3}(3k+2)$

$= \frac{1}{3}(k+1)(k^2+3k+2)$

$= \frac{1}{3}(k+1)(k+2)(k+1)$

$= \frac{1}{3}(k+1)^2(k+2)$ (1)

So if it is true for $n=k$, then true for $n=k+1$. Since true for $n=1$ then true for $n=2$ etc ... Induction true for all n.

(b) $y = 2\sin^{-1}\frac{x}{4}$

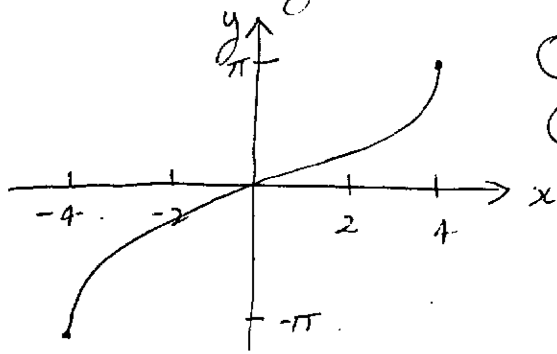
(i) Domain $-1 \le \frac{x}{4} \le 1$ (1)

then $-4 \le x \le 4$

TION 3 continued

Range: $-\frac{\pi}{2} \leq \sin^{-1}\left(\frac{x}{4}\right) \leq \frac{\pi}{2}$

$-\pi \leq y \leq \pi$



(1) for shape
(1) for domain & range.

(iii) $y = 2\sin^{-1}\frac{x}{4}$

$y' = \frac{2}{\sqrt{16-x^2}}$ ← (1)

At $y = \frac{\pi}{3}$, $\frac{\pi}{3} = 2\sin^{-1}\left(\frac{x}{4}\right)$

$\therefore \sin\frac{\pi}{6} = \frac{x}{4}$ So $x = 2$

$m = \frac{2}{\sqrt{16-4}} = \frac{1}{\sqrt{3}}$ ← (1)

(b)(i) Let $\sqrt{3}\cos x + \sin x = k\cos(x-\alpha)$

$\therefore \sqrt{3}\cos x + \sin x = (k\cos\alpha)\cos x + (k\sin\alpha)\sin x$

$\therefore k\cos\alpha = \sqrt{3}$ & $k\sin\alpha = 1$

Squaring & adding we get

$k^2\cos^2\alpha + k^2\sin^2\alpha = 3+1$

$\therefore k^2 = 4$ ← (1)

So $k = 2$ $(k>0.)$

$\therefore \cos\alpha = \frac{\sqrt{3}}{2}$ & $\sin\alpha = \frac{1}{2}$

$\therefore \alpha = \frac{\pi}{6}$ ← (1)

$\therefore \sqrt{3}\cos x + \sin x = 2\cos\left(x - \frac{\pi}{6}\right)$

(ii) $\sqrt{3}\cos x + \sin x = 1$ $0 \leq x \leq 2\pi$

$2\cos\left(x - \frac{\pi}{6}\right) = 1$

$\cos\left(x - \frac{\pi}{6}\right) = \frac{1}{2}$ (4th & 1st quad) ← (1)

then $x - \frac{\pi}{6} = \frac{\pi}{3}$ & $\frac{5\pi}{3}$

$\therefore x = \frac{\pi}{2}$ or $\frac{11\pi}{6}$. ← (1)

TOTAL 12 MARKS

QUESTION 4 -

(a)(i) $y = \frac{x^2}{4}$

$y' = \frac{2x}{4} = \frac{x}{2}$

m at $P(2t, t^2)$ is $\frac{2t}{2} = t$ ← (1)

$\therefore m$ of normal is $-\frac{1}{t}$ ← (1)

(ii) The normal has equation

$y - t^2 = -\frac{1}{t}(x - 2t)$

STION 4 continued

$ty - t^3 = -x + 2t$

$x + ty = t^3 + 2t$ ← 1

(iii) $(2,5)$ satisfies eqn. of normal.

$2 + t(5) = t^3 + 2t$

$t^3 - 3t - 2 = 0$ ← 1

(iv) By trial & error, $t = 2$ is a soln of the eqn in (iii)

∴ $(t-2)$ is a factor of ← 1

$t^3 - 3t + 2$

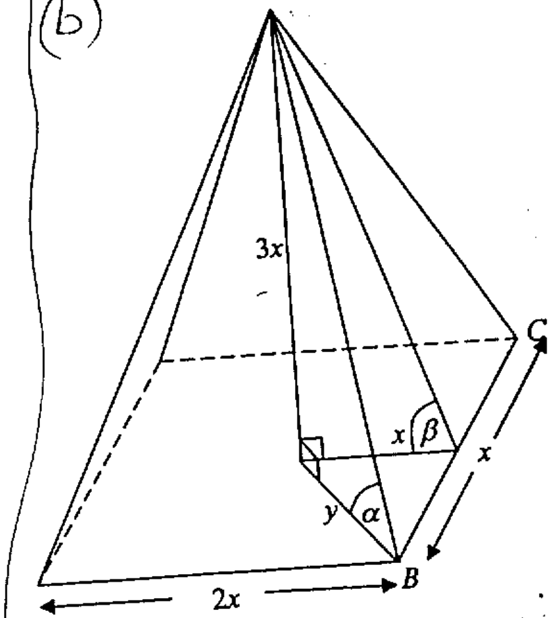
$$
\begin{array}{r}
t^2 + 2t + 1 \\
(t-2)\,\overline{)\,t^3 + 0t^2 - 3t - 2} \\
\underline{t^3 - 2t^2} \\
2t^2 - 3t \\
\underline{2t^2 - 4t} \\
t - 2 \\
\underline{t - 2} \\
0
\end{array}
$$
← 1

So eqn is $(t-2)(t+1)^2 = 0$ ← 1

∴ $t = -1$ or 2

So the two points are $(-2,1)$ & $(4,4)$ ← 1

(b)



Let the required angle be α

By Pyth. Thm

$y^2 = x^2 + \frac{1}{4}x^2$

$y^2 = \frac{5}{4}x^2$

$y = \frac{\sqrt{5}}{2}x$ — 1

∴ $\tan\alpha = \frac{3x}{y}$

$= 3x \div \frac{\sqrt{5}x}{2}$

$= 3x \times \frac{2}{\sqrt{5}x}$

1 → $= \frac{6}{\sqrt{5}}$

1 → ∴ $\alpha = 69°34'$

(ii) Let the required angle be β

$\tan\beta = \frac{3x}{x} = 3$

$\beta = 71°34'$ — 1

STION 5: continued.

(v) The particle starts from the origin and goes in a positive direction (+1m/s) Acceleration is always positive, so velocity is always positive. Hence the particle will continue to move in a positive direction, reaching a limiting velocity of 2m/s and never returning to the origin 1 1

TOTAL: 12 MARKS

QUESTION 6.

(a)(i) $V = \pi \int_0^{\frac{\pi}{2}} x^2 \, dy$

$y = \cos^{-1} x \quad \therefore \; x = \cos y$

$x^2 = \cos^2 y$

$V = \pi \int_0^{\frac{\pi}{2}} \cos^2 y \, dy$ ← 1

(ii) $V = \pi \int_0^{\frac{\pi}{2}} \cos^2 y \, dy$

since $\cos 2y = 2\cos^2 y - 1$

then $\cos^2 y = \frac{1}{2}(1 + \cos 2y)$ ← 1

$V = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} (1 + \cos 2y) \, dy$

$= \frac{\pi}{2} \left[y + \frac{1}{2} \sin 2y \right]_0^{\frac{\pi}{2}}$ ← 1

$= \frac{\pi}{2} \left[\left(\frac{\pi}{2} + 0\right) - (0 + 0) \right]$

$= \frac{\pi^2}{4} u^3$ ← 1

(b) $g(x) = 20 \log_e x - x^2$

$g'(x) = 20 \cdot \frac{1}{x} - 2x$

$= \frac{20}{x} - 2x$ ← 1

Put $x_1 = 5$ So $g(5) = 20 \log_e 5 - 25$, $g'(5) = 4 - 10 = -6$ ← 1

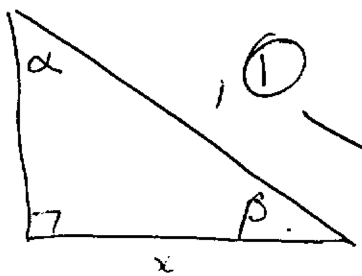
$x_2 = x_1 - \frac{g(x_1)}{g'(x_2)}$ where x_1 is appr

$= 5 - \frac{20 \log_e 5 - 25}{-6}$

$= 6.198126\ldots$

to four sign. figures, $x_2 = 6.198$ ← 1

(c)



1

$\sin \alpha = x \quad \cos \beta = x$ 1

$\alpha = \sin^{-1} x \quad \beta = \cos^{-1} x$

$\alpha + \beta = \frac{\pi}{2}$ (∠ sum of △

$\therefore \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

STION 6 CONTINUED

(c) $\int_{\frac{1}{2}}^{\frac{1}{3}} \sin^{-1}x + \cos^{-1}x \, dx = \int_{\frac{1}{2}}^{\frac{1}{3}} \frac{\pi}{2} dx$ from above

$= \left[\frac{\pi x}{2} \right]_{-\frac{1}{2}}^{\frac{1}{3}}$

$= \frac{\pi}{6} - -\frac{\pi}{4}$

$= \frac{5\pi}{12} \longleftarrow 1$

(d) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\Delta} = \frac{\alpha\beta\gamma + \alpha\beta\Delta + \alpha\beta\gamma + \beta\gamma\Delta}{\alpha\beta\gamma\Delta}$

$\boxed{\text{But } \alpha\beta\gamma\Delta = \frac{e}{a} = \frac{-2}{1}, \quad 1 \longrightarrow = -2}$

$= \frac{-d}{a} \div -2$

$= -1 \div -2$

$= \frac{1}{2} \longleftarrow \frac{1}{2}$

$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\Delta} = \frac{1}{2}$

$\alpha = 1$ so $1 + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\Delta} = \frac{1}{2}$

$\therefore \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\Delta} = -\frac{1}{2} \longleftarrow \frac{1}{2}$

TOTAL 12 MARK.

QUESTION 7:

(a) $\frac{d(\frac{1}{2}v^2)}{dx} = \frac{1}{8}(x^3 + x)$

$\frac{1}{2}v^2 = \frac{1}{8}\int (x^3 + x) dx$

$= \frac{1}{8}\left[\frac{x^4}{4} + \frac{x^2}{2} + C\right]$

$v^2 = \frac{1}{8}\left[\frac{x^4}{4} + \frac{x^2}{2}\right] + C \longleftarrow 1$

when $v = \frac{1}{4}$, $x = 0$

$\frac{1}{2} \times \frac{1}{16} = \frac{1}{8}\left[\frac{0}{4} + \frac{0}{2}\right] + \boxed{C}$

$\therefore \frac{1}{32} = C \longleftarrow 1$

$\therefore \frac{1}{2}v^2 = \frac{x^4}{32} + \frac{x^2}{16} + \frac{1}{32}$

$= \frac{1}{16}(x^4 + 2x^2 + 1)$

$= \left(\frac{x^2 + 1}{16}\right)^2 \longleftarrow 1$

(b) (i) Length of arc PQ

$= r x$ $\boxed{1}$

In Δ OQR

$OQ = OP = r$

$\therefore QR = OQ \sin x$

$= r \sin x$ $\boxed{1}$

(ii) $\overset{\frown}{A_0B_0} = x$

$A_1B_0 = \sin x$

$OA_1 = \cos x$

$\therefore$ In sector OA_1B_1, $r = \cos x$

$\therefore$ $\overset{\frown}{A_1B_1} = x\cos x$

$A_2B_1 = \sin x \cos x$ using (i) of (b)

Similarly

$\overset{\frown}{A_2B_2} = x\cos^2 x$

$A_3B_2 = \sin x \cos^2 x$ putting $r = \cos^2 x$ in (b)(i)

. . . .

[1]

$\therefore$ Total distance travelled

$$y = \overset{\frown}{A_0B_0} + B_0A_1 + \overset{\frown}{A_1B_1} + B_1A_2 + \overset{\frown}{A_2B_2} + B_2A_3 + \cdots$$

$$= x + \sin x + x\cos x + \sin x\cos x + x\cos^2 x + \sin x \cos^2 x + \cdots$$

$$= x(1+\cos x+\cos^2 x+\cdots) + \sin x(1+\cos x+\cos^2 x+\cdots)$$

$$= (x+\sin x)(1+\cos x+\cos^2 x+\cdots)$$ [1]

$$= (x+\sin x)\cdot\frac{1}{1-\cos x}$$ [1]

since y is an infinite G.P with $r = \cos x$
and $\underline{-1<\cos x<1}$ } [1]

(iii) $$y' = \frac{(1+\cos x)(1-\cos x)-(\sin x)(x+\sin x)}{(1-\cos x)^2}$$

$$= \frac{1-\cos^2 x - x\sin x - \sin^2 x}{(1-\cos x)^2}$$

$$= \frac{-x\sin x}{(1-\cos x)^2}$$ [1]

$\therefore$ $y' < 0$ since $x>0$, $\sin x<0$ $0<x<\frac{\pi}{2}$
and $(1-\cos x)^2>0$

Hence y is a decreasing function [1]

$\therefore$ min value occurs when $x = \frac{\pi}{2}$ [1]