CSSA 2023 Advanced Mathematics Solutions

by Ty Webb

1. A 2. B 3. A 4. C 5. D 6. C 7. B 8. D 9. C 10. B

11. $77 - 34 = 43\%$ although answers may vary slightly

12. $y' = 3x^2 - 2 \therefore -\frac{1}{y'(-1)} = -1 \therefore y - 1 = -(x+1) \therefore y = -x$

13. $\frac{4\times5}{8} = 2.5\text{hours}$

14a. $500 \times 44.11376 = \$22056.88$

14b. $22056.88(1.0035)^{48} = \$26084.24$

14c. $46 \times \frac{49000}{41.00219} - 49000 = \$5972.67$

15a. $\frac{2\ln x}{x}$

15b. $\frac{1}{2}[(\ln x)^2]_1^e = \frac{1}{2}$

16. $F(x) = \int(\sec^2 x - 1)\,dx = \tan x - x + c$ and $0 = \tan\frac{\pi}{3} - \frac{\pi}{3} + c$
so $c = \frac{\pi}{3} - \sqrt{3}$ and $F(x) = \tan x - x + \frac{\pi}{3} - \sqrt{3}$

17a. $f(x) = \sin(\frac{\pi}{2} - \frac{x}{2}) = \cos\frac{x}{2}$ and so

$$\begin{aligned} f(-x) &= -\sin(\tfrac{1}{2}(-x-\pi)) \\ &= \sin(\tfrac{1}{2}(x+\pi)) \\ &= \cos(\tfrac{\pi}{2} - \tfrac{1}{2}(x+\pi)) \\ &= \cos -\tfrac{x}{2} \\ &= \cos\tfrac{x}{2} \\ &= f(x) \therefore \text{ even} \end{aligned}$$

17b. $2\int_0^{\pi}\cos\frac{x}{2}\,dx = [4\sin\frac{x}{2}]_0^{\pi} = 4 - 0 = 4$

18a. $1 - 0.75 - 0.03 - 0.002 - 0.001 - 0.0001 = 0.2169$

18b. Expected to lose
$20 - 0(0.75) - 20(0.2169) - 100(0.03) - 500(0.002) - 5000(0.001) - 10000(0.0001)$
$= \$5.66$

18c.
$0^2(0.75)+20^2(0.2169)+100^2(0.03)+500^2(0.002)+5000^2(0.001)+10000^2(0.0001)$
$-(0(0.75)+20(0.2169)+100(0.03)+500(0.002)+5000(0.001)+10000(0.0001))^2$
$\approx \$35681.1818$

19a. 25°C

19b. $\frac{175-150\times0.9^{0.2\times75}-(175-150\times0.9^{0.2\times25})}{75-25} = 1.15°\text{C per minute}$

19c. $-150\times\ln 0.9\times 0.2\times 0.9^{0.2t}\leq 1 \therefore 0.9^{0.2t}\leq \frac{1}{-30\ln 0.9}$
$\therefore t\geq 5\log_{0.9}\frac{1}{-30\ln 0.9}\approx 55\text{min} \therefore$ 11:25am

20. $A=0.6(A+32) \therefore 0.4A=19.2 \therefore A=48 \therefore P(CD)=\frac{48}{48+45+32+35}=0.3=30\%$

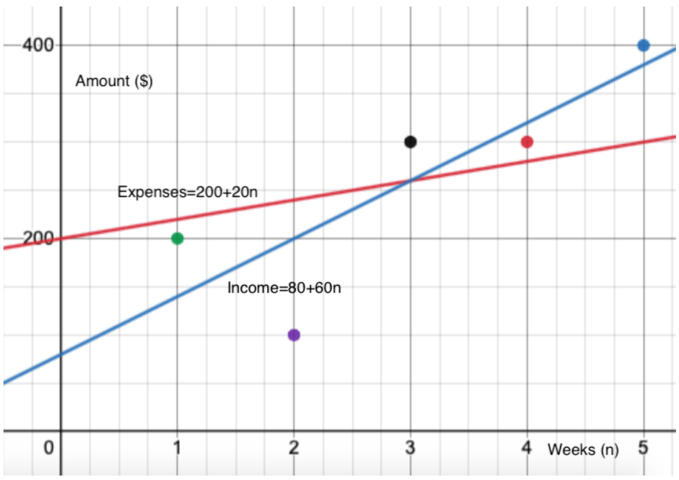
21. $20\log_{10}\frac{N}{N_0}=8 \therefore \frac{N}{N_0}=10^{\frac{8}{20}}\approx 2.5$

22a. $P(z>-1.2)=P(z<1.2)=0.885$

22b. $\frac{599-602}{2.5}=-1.2$ and $\frac{604-602}{2.5}=0.8$

$$\begin{aligned}\therefore P(-1.2<z<0.8) &= P(z<0.8)+P(z<1.2)-1\\ &= 0.788+0.885-1\\ &= 0.673\\ &= 67.3\%\end{aligned}$$

23. Income $=80+60n$ and Expenses $=200+20n$



24a. $\frac{1}{2}(10+1.77+2(3.68+2.43))=11.995$

24b. As the curve is concave up trapezoidal rule gives an overestimate and so the maximum will also be overestimated. The safety valve will be shut off before it reaches the maximum which is lower than the calculated maximum

25a. $A = \pi r^2 + 2rh + \frac{1}{2}\pi(2r)h = \pi r^2 + (2r + \pi r)\frac{2V}{\pi r^2} = \pi r^2 + \frac{2V}{\pi} \cdot \frac{2+\pi}{r}$

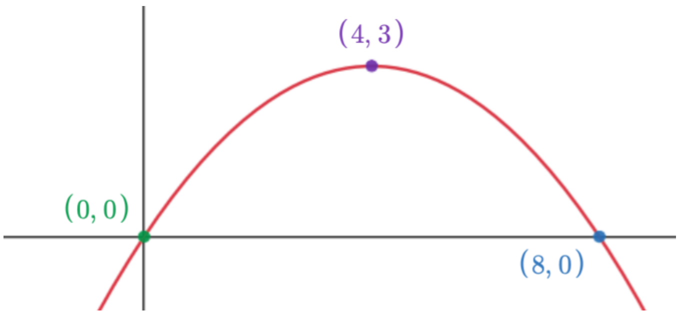
25b. $\frac{dA}{dr} = 2\pi r + \frac{2V}{\pi}(2+\pi)(-r^{-2}) = 0$ to minimise A
$\therefore r^3 = \frac{V(2+\pi)}{\pi^2} \therefore r = \sqrt[3]{\frac{V(2+\pi)}{\pi^2}}$
Also $\frac{d^2A}{dr^2}\Big|_{r=\sqrt[3]{\frac{V(2+\pi)}{\pi^2}}} = 2\pi + \frac{4V(2+\pi)}{\pi r^3}\Big|_{r=\sqrt[3]{\frac{V(2+\pi)}{\pi^2}}} = 6\pi > 0$
$\therefore r = \sqrt[3]{\frac{V(2+\pi)}{\pi^2}}$ minimises A

25c. $\frac{1}{2}\pi\left(\sqrt[3]{\frac{10(2+\pi)}{\pi^2}}\right)^2 h = 10 \Rightarrow h = 20\sqrt[3]{\frac{\pi}{100(2+\pi)^2}} \approx 2.118\text{m}$

26a. $120° - 45° = 75°$ and so $\frac{1}{2} \times 100 \times \text{MB} \times \sin 75° = 10432$
$\therefore \text{MB} = 208.64 \csc 75° \approx 216\text{m}$

26b. $\text{AT} = p \Rightarrow \text{MT}^2 + 100^2 = p^2$ and $\text{MT}^2 + (208.64 \csc 75°)^2 = (380-p)^2$
$\therefore \text{MT}^2 = p^2 - 100^2 = (380-p)^2 - (208.64\csc 75°)^2 \therefore p = \frac{100^2+380^2-(208.64\csc 75°)^2}{760}$
$\therefore \text{MT} = \sqrt{\left(\frac{100^2+380^2-(208.64\csc 75°)^2}{760}\right)^2 - 100^2} \approx 100\text{m}$

27. $f_1(x) = (x+1)^2$ shift right 4 $\Rightarrow$ replace x by $x-4$
$\Rightarrow f_2(x) = (x-3)^2$
Stretch horizontally by factor $\frac{4}{3} \Rightarrow$ replace x with $\frac{x}{4/3}$
$\Rightarrow f_3(x) = (\frac{3x}{4} - 3)^2$
Stretch vertically by factor $\frac{1}{3} \Rightarrow$ replace $f_3(x)$ with $\frac{f_3(x)}{1/3}$
$\Rightarrow 3f_4(x) = (\frac{3x}{4}-3)^2 \Rightarrow f_4(x) = \frac{1}{3}(\frac{3x}{4}-3)^2$
Shift down by 3 $\Rightarrow$ replace $f_4(x)$ with $f_4(x) + 3 \Rightarrow$
$f_5(x) + 3 = \frac{1}{3}(\frac{3x}{4}-3)^2 \Rightarrow f_5(x) = -3 + \frac{1}{3}(\frac{3x}{4}-3)^2$
Reflect about x-axis $\Rightarrow$ replace $f_5(x)$ with $-f_5(x)$
$\Rightarrow -f_6(x) = -3 + \frac{1}{3}(\frac{3x}{4}-3)^2$
$\Rightarrow f_6(x) = 3 - \frac{1}{3}(\frac{3x}{4}-3)^2 = 3 - \frac{3}{16}(x-4)^2 = \frac{3}{16}x(x-8)$
$\therefore x$-intercepts are $0, 8$ and vertex $= (4,3)$



28a. $\dot{x} = \frac{4t}{t^2+3} - 1 = 0$ when at rest $\Rightarrow t^2 + 3 = 4t \therefore (t-1)(t-3) = 0$ and $t = 1, 3$

28b. $|x(1)-x(0)| = |2\ln 4 - 1 - 2\ln 3| = \ln\frac{9}{16}+1 = (\ln\frac{9}{16} - -1)\text{m}$

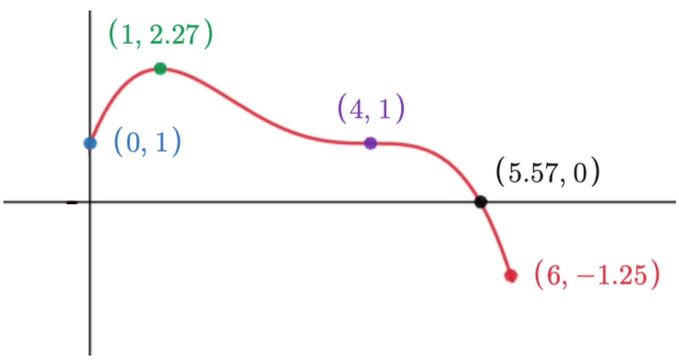
Note that the last step seems unnecessary, however it was stipulated in the question to express it in the form $\ln\frac{a}{b} - c$

28c. $\ddot{x} = \frac{(t^2+3)(4)-4t(2t)}{(t^2+3)^2} = \frac{12-4t^2}{(t^2+3)^2} = 0$ for maximum $\dot{x}$
$\therefore t = \sqrt{3}$ and $\dot{x}(\sqrt{3}) = \frac{4\sqrt{3}}{6} - 1 = \frac{2\sqrt{3}-3}{3}$
Also $\ddot{x}|_{t=\sqrt{3}} = \frac{(t^2+3)^2(-8t)-(12-4t^2)(2)(t^2+3)(2t)}{(t^2+3)^4}|_{t=\sqrt{3}} = -\frac{2\sqrt{3}}{9} < 0$
$\therefore$ maximum $\dot{x} = \frac{2\sqrt{3}-3}{3}\text{ms}^{-1}$

29a. $f'(x) = 3x(3)(1-\frac{x}{4})^2(-\frac{1}{4}) + (1-\frac{x}{4})^3(3)$
$= 3(1-\frac{x}{4})^2(-\frac{3x}{4}+1-\frac{x}{4})$
$= 3(1-\frac{x}{4})^2(1-x)$
$= 0$ for stationary points
$\therefore x = 1, 4$ and $f(1) = 3(\frac{27}{64})+1 = \frac{145}{64} \approx 2.27, f(4) = 1$
Hence the stationary points are $(1, \frac{145}{64}), (4, 1)$
Also, $f''(x) = 3(1-\frac{x}{4})^2(-1) + 3(1-x)(2)(1-\frac{x}{4})(-\frac{1}{4})$
$= \frac{9}{4}(1-\frac{x}{4})(x-2)$
$\therefore f''(1) = -\frac{27}{16} < 0 \therefore (1, \frac{145}{64})$ is a local maximum turning point
Also, $f''(4) = 0$
$f'''(x) = \frac{9}{4}(1-\frac{x}{4})(1) + \frac{9}{4}(x-2)(-\frac{1}{4})$
$= -\frac{9}{8}(x-3)$
$\therefore f'''(4) = -\frac{9}{8} \neq 0$ and so $(4, 1)$ is a horizontal inflection point

29b. $f(5) = 15(-\frac{1}{4})^3 + 1 = \frac{49}{64} > 0$ and $f(6) = 18(-\frac{1}{2})^3 + 1 = -\frac{5}{4} < 0$
As $f(x)$ changes sign between $x = 5$ and $x = 6$ and $f(x)$ is a continuous function, it will have at least 1 x-intercept between $x = 5$ and $x = 6$

29c. y-intercept is $y = 1$
x-intercept is $x = \frac{6-\sqrt{\alpha-12}+\sqrt{2\sqrt{\alpha^2+\frac{256}{3}}-\alpha-12\sqrt{\alpha-12}+24}}{2}$
where $\alpha = \frac{8}{3}(6 - \sqrt[3]{\sqrt{145}+9} + \sqrt[3]{\sqrt{145}-9})$
Hence $x \approx 5.57$



$30a.\ A_1 = 500000(1.03) - 25000$

$$\begin{aligned} A_2 &= A_1(1.03) - 25000(1.05) \\ &= 500000(1.03)^2 - 25000(1.03) - 25000(1.05) \\ A_3 &= A_2(1.03) - 25000(1.05)^2 \\ &= 500000(1.03)^3 - 25000(1.03)^2 - 25000(1.03)(1.05) - 25000(1.05)^2 \\ &= 500000(1.03)^3 - 25000((1.05)^2 + (1.05)(1.03) + (1.03)^2) \end{aligned}$$

$$\begin{aligned} 30b.\ A_n &= 500000(1.03)^n - 25000((1.05)^{n-1} + (1.05)^{n-2}(1.03) + \cdots + (1.03)^{n-1}) \\ &= (1.03)^{n-1}(500000(1.03) - 25000((\tfrac{1.05}{1.03})^{n-1} + (\tfrac{1.05}{1.03})^{n-2} + \cdots + 1)) \\ &= 5000(1.03)^{n-1}\left(103 - 5\frac{(\frac{1.05}{1.03})^n - 1}{\frac{1.05}{1.03} - 1}\right) \\ &= 257500(1.03)^{n-1}(7 - 5(\tfrac{1.05}{1.03})^n) \\ &= 0 \text{ when paid off} \end{aligned}$$

$\therefore (\frac{1.05}{1.03})^n = 1.4 \therefore n = \log_{\frac{1.05}{1.03}} 1.4 \approx 17$

$31a.\ \int_1^2 k(t-1)^2\,dt + \int_2^4 k(2 - \frac{t}{2})\,dt = 1$

$\therefore [\frac{(t-1)^3}{3}]_1^2 + [2t - \frac{t^2}{4}]_2^4 = \frac{1}{k}$

$\therefore \frac{1}{3} - 0 + 8 - 4 - (4 - 1) = \frac{4}{3} = \frac{1}{k}$

$\therefore k = \frac{3}{4}$

$31b.\ \int_1^2 \frac{3}{4}(t-1)^2\,dt + \int_2^T \frac{3}{4}(2 - \frac{t}{2})\,dt = 0.88$

$$\begin{aligned} \therefore \tfrac{1}{4}[(t-1)^3]_1^2 + \tfrac{3}{4}[2t - \tfrac{t^2}{4}]_2^T &= \tfrac{1}{4} + \tfrac{3}{4}(2T - \tfrac{T^2}{4} - (4-1)) \\ &= -2 + \tfrac{3T}{2} - \tfrac{3T^2}{16} \\ &= 0.88 \end{aligned}$$

$\therefore -32 + 24T - 3T^2 = 14.08 \therefore 3T^2 - 24T + 46.08 = (T - 3.2)(3T - 14.4) = 0$

$T \neq \frac{14.4}{3} = 4.8 > 4 \therefore T = 3.2\text{min}$

$$\begin{aligned} 32.\ h'(x) &= 2f'(x)f(x) \\ &= 2(f'(g(t))g'(t))f(x) \\ \therefore h'(1) &= 2(f'(g(1))g'(1))f(3) \\ &= 2(f'(3)(4))(5) \\ &= 2(4 \times 2)(5) \\ &= 80 \end{aligned}$$