CSSA 2023 Mathematics Extension 1 Solutions

by Ty Webb

1. C 2. C 3. D 4. C 5. A 6. B 7. B 8. A 9. A 10. D

11a $-5 < 5x - 1 < 5 \therefore -4 < 5x < 6 \therefore -\frac{4}{5} < x < \frac{6}{5}$

11b. $\int \frac{1}{9+25x^2}\,dx = \frac{1}{5}\int \frac{5}{3^2+(5x)^2}\,dx = \frac{1}{5}\cdot\frac{1}{3}\cdot\tan^{-1}\frac{5x}{3} + c = \frac{1}{15}\tan^{-1}\frac{5x}{3} + c$

11ci. $-\frac{3}{1} = -3$

11cii. $(\sum\alpha)^2 - 2\sum\alpha\beta = (-3)^2 - 2(\frac{-4}{1}) = 17$

11di. $2\times 5 + 3\times(-12) = -26$

11dii. $\sqrt{5^2+12^2} = 13$

11$diii$. $\text{Proj}_{\underset{\sim}{u}}\ \underset{\sim}{v} = \frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{|\underset{\sim}{v}|^2}\underset{\sim}{v}$

$$= \frac{-26}{13^2}(5\underset{\sim}{i} - 12\underset{\sim}{j})$$
$$= \frac{-2}{13}(5\underset{\sim}{i} - 12\underset{\sim}{j})$$
$$= \frac{-10}{13}\underset{\sim}{i} + \frac{24}{13}\underset{\sim}{j}$$

11e. $\frac{10!}{3!} = 604800$

12a. $\int e^y\,dy = \int 2x^{-3}\,dx \therefore e^y = -x^{-2} + c$ and $e^0 = 1 = -(1)^{-2} + c = -1 + c \therefore c = 2$
$\therefore e^y = 2 - x^{-2} \therefore y = \ln(2 - x^{-2})$

12b. $\tan 2x = \tan(\frac{\pi}{4} + x - (\frac{\pi}{4} - x))$

$$= \frac{\tan(\frac{\pi}{4}+x)-\tan(\frac{\pi}{4}-x)}{1+\tan(\frac{\pi}{4}+x)\tan(\frac{\pi}{4}-x)}$$
$$= \frac{\tan(\frac{\pi}{4}+x)-\tan(\frac{\pi}{4}-x)}{1+\tan(\frac{\pi}{4}+x)\cot(\frac{\pi}{2}-(\frac{\pi}{4}-x))}$$
$$= \frac{\tan(\frac{\pi}{4}+x)-\tan(\frac{\pi}{4}-x)}{1+\tan(\frac{\pi}{4}+x)\cot(\frac{\pi}{4}+x)}$$
$$= \frac{\tan(\frac{\pi}{4}+x)-\tan(\frac{\pi}{4}-x)}{1+1}$$

$\therefore \tan(\frac{\pi}{4}+x) - \tan(\frac{\pi}{4}-x)) = 2\tan(2x)$

12c. $|3\underset{\sim}{a} - 2\underset{\sim}{b}|^2 = (3\underset{\sim}{a} - 2\underset{\sim}{b})\cdot(3\underset{\sim}{a} - 2\underset{\sim}{b})$

$$= 9\underset{\sim}{a}\cdot\underset{\sim}{a} - 12\underset{\sim}{a}\cdot\underset{\sim}{b} + 4\underset{\sim}{b}\cdot\underset{\sim}{b}$$
$$= 9|\underset{\sim}{a}|^2 - 12(5) + 4|\underset{\sim}{b}|^2$$
$$= 9(2)^2 - 60 + 4(3)^2$$
$$= 12$$

$\therefore |3\underset{\sim}{a} - 2\underset{\sim}{b}| = \sqrt{12} = 2\sqrt{3}$

$12d.\ \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (\sin x - \cos x)^2\,dx = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (\sin^2 x + \cos^2 x - 2\sin x\cos x)\,dx$

$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (1 - \sin 2x)\,dx$

$= [x + \frac{1}{2}\cos 2x]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$

$= \frac{3\pi}{4} + \frac{1}{2}\cos\frac{3\pi}{2} - (\frac{\pi}{4} + \frac{1}{2}\cos\frac{\pi}{2})$

$= \frac{\pi}{2}$

$12e.\ 23^1 - 1 = 22 = 2 \times 11 \therefore$ true for $n = 1$

If true for $n = k, 23^k - 1 = 11M$ for an integer M so $23^k = 11M + 1$

$\therefore 23^{k+1} - 1 = 23(23^k) - 1$

$= 23(11M + 1) - 1$

$= 11(23M) + 22$

$= 11(23M + 2)$

and since $23M + 2$ is an integer then it is true for $n = k + 1$

Hence by the principle of mathematical induction it is true for all positive integers n

$13ai.\ \frac{d}{dx}(\tan^3 x) = \frac{d}{dx}(\tan x \cdot \tan x \cdot \tan x)$

$= (\frac{d}{dx}\tan x) \cdot \tan x \cdot \tan x + \tan x \cdot (\frac{d}{dx}\tan x) \cdot \tan x + \tan x \cdot \tan x \cdot (\frac{d}{dx}\tan x)$

$= 3\tan^2 x \cdot (\frac{d}{dx}\tan x)$

$= 3(\sec^2 x - 1)\sec^2 x$

$= 3\sec^4 x - 3\sec^2 x$

$13aii.$ From $13ai, \sec^4 x = \sec^2 x + \frac{d}{dx}(\frac{1}{3}\tan^3 x)$

$\therefore \int_0^{\frac{\pi}{4}} \sec^4 x\,dx = \int_0^{\frac{\pi}{4}} (\sec^2 x + \frac{d}{dx}(\frac{1}{3}\tan^3 x))\,dx$

$= [\tan x + \frac{1}{3}\tan^3 x]_0^{\frac{\pi}{4}}$

$= 1 + \frac{1}{3} - (0 + 0)$

$= \frac{4}{3}$

$13b.\ \frac{3}{\pi}\tan^{-1}(\sqrt{a}) - \frac{3}{\pi}\tan^{-1}(\sqrt{0}) = 1 \therefore \tan^{-1}(\sqrt{a}) = \frac{\pi}{3} \therefore \sqrt{a} = \tan\frac{\pi}{3} = \sqrt{3} \therefore a = 3$

$13ci.\ 2\cos 3x\cos 2x = \cos(3x - 2x) + \cos(3x + 2x) \therefore \cos x + \cos 5x = 2\cos 3x\cos 2x$

$13cii.\ 2\cos 3x\cos 2x = \cos 2x, x \in [0, \pi]$ and so $\cos 2x(2\cos 3x - 1) = 0$

$\therefore \cos 2x = 0$ or $\cos 3x = \frac{1}{2}$

$2x \in [0, 2\pi]$ and $3x \in [0.3\pi]$

$2x = \frac{\pi}{2}, \frac{3\pi}{2}$ or $3x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3} \therefore x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$

$13d.$ When $x = 0, u = 0$ and when $x = \sqrt{3}, u = \frac{\pi}{3}$

$x = \tan u$ and $dx = \sec^2 u\,du$ and $1 + x^2 = 1 + \tan^2 u = \sec^2 u$

$$\int_0^{\sqrt{3}} \frac{1}{(1+x^2)^{3/2}}\,dx = \int_0^{\frac{\pi}{3}} \frac{1}{(\sec^2 u)^{3/2}} \sec^2 u\,du$$
$$= \int_0^{\frac{\pi}{3}} \cos u\,du$$
$$= [\sin u]_0^{\frac{\pi}{3}}$$
$$= \frac{\sqrt{3}}{2}$$

13e. $x = e^y + 1$ and the horizontal line is $y = \log_e(k-1)$

$\therefore \int_0^{\log_e(k-1)} \pi x^2\,dy = \frac{\pi(27+\log_e 16)}{2}$

$\therefore \int_0^{\log_e(k-1)} (e^{2y} + 2e^y + 1)\,dy = [\frac{1}{2}e^{2y} + 2e^y + y]_0^{\log_e(k-1)}$

$$= \frac{1}{2}(k-1)^2 + 2(k-1) + \log_e(k-1) - (\frac{1}{2} + 2 + 0)$$
$$= \frac{k^2+2k-8}{2} + \log_e(k-1)$$
$$= \frac{(k-5)(k+7)+27}{2} + \log_e(k-1)$$
$$= \frac{27+\log_e 16}{2}$$
$$= \frac{27}{2} + \log_e 4$$

Since $k > 2, k = 5$

14a. $z = \frac{10-10.25}{0.5} = -0.5$ and so $P(Z > -0.5) = P(Z < 0.5) = 0.6915$

$\sum_{n=13}^{25} \binom{25}{n} 0.6915^n \times 0.3085^{25-n} \approx 0.977874 \approx 98\%$

More accurately using the complementary error function instead of the table,

$\sum_{n=13}^{25} \binom{25}{n} \left(1 - 0.5\text{erfc}\left(\frac{0.5}{\sqrt{2}}\right)\right)^n \times \left(0.5\text{erfc}\left(\frac{0.5}{\sqrt{2}}\right)\right)^{25-n} \approx 0.977851 \approx 98\%$

Alternatively using the regularised incomplete beta function,

$1 - 13\binom{25}{12} \int_0^{0.3085} t^{12}(1-t)^{12}\,dt \approx 0.977874 \approx 98\%$

and more accurately with the complementary error function,

$1 - 13\binom{25}{12} \int_0^{0.5\text{erfc}\left(\frac{0.5}{\sqrt{2}}\right)} t^{12}(1-t)^{12}\,dt \approx 0.977851 \approx 98\%$

14b. $V = Ah \therefore \frac{dV}{dt} = \frac{d}{dh}(Ah) \cdot \frac{dh}{dt} = A \cdot \frac{dh}{dt} = -h\sqrt{h} \Rightarrow \frac{dh}{dt} = -\frac{h^{\frac{3}{2}}}{A} \Rightarrow \int -h^{-\frac{3}{2}}\,dh = \int \frac{1}{A}\,dt$

$2h^{-\frac{1}{2}} = \frac{t}{A} + c$

Now $A = 10, t = 0, h = 4 \Rightarrow c = 1 \therefore 2h^{-\frac{1}{2}} = 1 + \frac{t}{10}$

$\therefore t = 10(2h^{-\frac{1}{2}} - 1)$ and $V = 10h = 1\text{L} = 0.001\text{m}^3 \Rightarrow h = \frac{1}{10000} \therefore t = 1990$ hours

14ci. $10\sqrt{2}t = 10 \Rightarrow t = \frac{1}{\sqrt{2}} \therefore y = 10\sqrt{2} \cdot \frac{1}{\sqrt{2}} - 5(\frac{1}{2}) + 1 = \frac{17}{2}$metres

14cii. $\begin{pmatrix} 10\sqrt{2} \\ 10\sqrt{2}-10t \end{pmatrix}$

14$ciii$. $U = \sqrt{(10\sqrt{2})^2 + (10\sqrt{2} - \frac{10}{\sqrt{2}})^2} = 5\sqrt{10}\text{m/s}$, $\beta = \cos^{-1} \frac{10\sqrt{2}}{5\sqrt{10}} = 26°34'$

$14civ$. At t seconds after hitting the wall $\underset{\sim}{a} = \binom{0}{-10}$
$\underset{\sim}{v} = \int \underset{\sim}{a}(t)\,dt = \int \binom{0}{-10}\,dt = \binom{c_1}{c_2 - 10t}$
$t = 0 \Rightarrow c_1 = \frac{-10\sqrt{2}}{\sqrt{2}} = -10, c_2 = (10\sqrt{2} - \frac{10}{\sqrt{2}}) \cdot \frac{1}{\sqrt{2}} = 5$
$\Rightarrow \underset{\sim}{v}(t) = \binom{-10}{5 - 10t}$
$\underset{\sim}{r}(t) = \int \underset{\sim}{v}(t)\,dt = \int \binom{-10}{5 - 10t}\,dt = \binom{c_3 - 10t}{5t - 5t^2 + c_4}$
At $P, t = 0, c_3 = 10, c_4 = \frac{17}{2} \Rightarrow \underset{\sim}{r}(t) = \binom{10 - 10t}{5t - 5t^2 + \frac{17}{2}}$
At $Q, 10 - 10t = 0 \therefore t = 1 \Rightarrow 5t - 5t^2 + \frac{17}{2} = \frac{17}{2}$, same height as P