CSSA 2023 Mathematics Extension 2 Solutions

by Ty Webb

1. C 2. B 3. A 4. D. 5. A 6. D 7. C 8. A 9. B 10. C

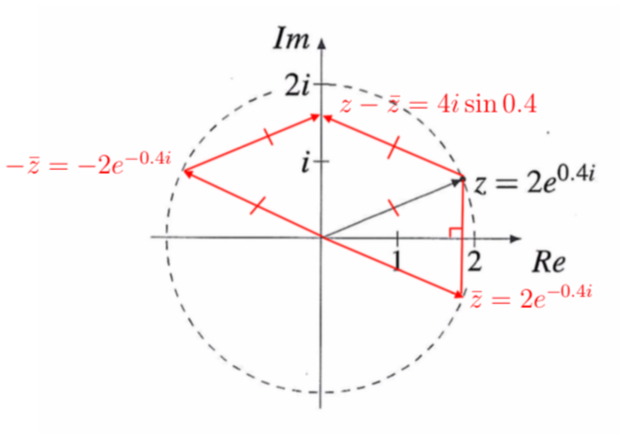
11a. $2\sqrt{2}e^{-\frac{3\pi}{4}i} = 2\sqrt{2}\cos -\frac{3\pi}{4} + 2\sqrt{2}i\sin -\frac{3\pi}{4} = 2\sqrt{2}(-\frac{1}{\sqrt{2}}) - 2\sqrt{2}i(\frac{1}{\sqrt{2}}) = -2-2i$

11bi. $\begin{pmatrix} 2-2 \\ -2-2 \\ 2-2 \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix}$

11bii. $\sqrt{0^2+4^2+0^2} = 4$

11$biii$. $\cos^{-1}\frac{\overrightarrow{OA}\cdot\overrightarrow{OB}}{|\overrightarrow{OA}||\overrightarrow{OB}|} = \cos^{-1}\frac{2(2)+2(-2)+2(2)}{\sqrt{2^2+2^2+2^2}\sqrt{2^2+2^2+2^2}} = \cos^{-1}\frac{1}{3} \approx 71°$

11c.



11d. $\ddot{x} = -4^2x \Rightarrow$ S.M.H. with period $\frac{2\pi}{4}$ $\therefore$ it first returns to origin in $\frac{\pi}{4}$s.

11e. If $z = r_1e^{i\theta_1}$ and $w = r_2e^{i\theta_1}$

$$\begin{aligned}\overline{zw} &= \overline{r_1e^{i\theta_1}r_2e^{i\theta_2}} \\ &= r_1r_2\overline{e^{i(\theta_1+\theta_2)}} \\ &= r_1r_2e^{-i(\theta_1+\theta_2)} \\ &= r_1e^{-i\theta_1}r_2e^{-i\theta_2} \\ &= \bar{z}\bar{w}\end{aligned}$$

12ai $-1+2\mu = 9 \therefore \mu = 5 \therefore k = 2+5(-3) = -13$

12aii Point of intersection is (x,y,z) where $\frac{x-7}{1} = \frac{y+1}{3} = \frac{z-2}{-5}$ and $\frac{x+1}{2} = \frac{y-2}{-3} = \frac{z+6}{6}$ which are 6 planes. Choosing 3 of these $3x - y + 0z = 22, -5x + 0y - z = -37$ and $0x + 2y + z = -2$ we have $\begin{pmatrix} 3 & -1 & 0 \\ -5 & 0 & -1 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 22 \\ -37 \\ -2 \end{pmatrix}$

$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & -1 & 0 \\ -5 & 0 & -1 \\ 0 & 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 22 \\ -37 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ -7 \\ 12 \end{pmatrix}$ and so the intersection is $(5, -7, 12)$

$12b.$ Let $u^2 = 1 - x^2 \therefore$ when $x = 0, u = 1$ and when $x = \frac{1}{\sqrt{2}}, u = \frac{1}{\sqrt{2}}$

Also $2u\,du = -2x\,dx$ and $x^2 = 1 - u^2$ and $\frac{x\,dx}{\sqrt{1-x^2}} = -du$

$$\begin{aligned}\therefore \int_0^{\frac{1}{\sqrt{2}}} \frac{x^3}{\sqrt{1-x^2}}\,dx &= \int_1^{\frac{1}{\sqrt{2}}} -(1-u^2)\,du \\ &= \int_{\frac{1}{\sqrt{2}}}^1 (1-u^2)\,du \\ &= [u - \tfrac{u^3}{3}]_{\frac{1}{\sqrt{2}}}^1 \\ &= 1 - \tfrac{1}{3} - (\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{12}) \\ &= \tfrac{8-5\sqrt{2}}{12}\end{aligned}$$

$$\begin{aligned}12c.\ z^2 + (7-i)z + 16 + 4i &= z^2 + (7-i)z + \tfrac{(7-i)^2}{4} + 16 + 4i - \tfrac{(7-i)^2}{4} \\ &= (z + \tfrac{7-i}{2})^2 + \tfrac{16+30i}{4} \\ &= (z + \tfrac{7-i}{2})^2 - (\tfrac{i\sqrt{16+30i}}{2})^2 \\ &= (z + \tfrac{7-i}{2})^2 - (\tfrac{i\sqrt{25-9+2\times5\times3i}}{2})^2 \\ &= (z + \tfrac{7-i}{2})^2 - \left(\frac{i\sqrt{(5+3i)^2}}{2}\right)^2 \\ &= (z + \tfrac{7-i}{2})^2 - \left(\tfrac{5i-3}{2}\right)^2 \\ &= (z + \tfrac{7-i+5i-3}{2})(z + \tfrac{7-i-5i+3}{2}) \\ &= (z + 2 + 2i)(z + 5 - 3i) \\ &= 0 \therefore z = -2 - 2i, -5 + 3i\end{aligned}$$

$$\begin{aligned}12d.\ \int \tfrac{x^3-2}{x^3+x}\,dx &= \int \tfrac{2x^2-x-2x^2-2+x^3+x}{(x^2+1)x}\,dx \\ &= \int (\tfrac{2x-1}{x^2+1} - \tfrac{2}{x} + 1)\,dx \\ &= \ln(x^2+1) - \tan^{-1} x - 2\ln|x| + x + c\end{aligned}$$

$$\begin{aligned}13a.\ (\sqrt{ab} - \tfrac{1}{\sqrt{ab}})^2 &= ab - 2 + \tfrac{1}{ab} \\ &= ab + 2 + \tfrac{1}{ab} - 4 \\ &= (a + \tfrac{1}{b})(b + \tfrac{1}{a}) - 4 \\ &\geq 0 \therefore (a + \tfrac{1}{b})(b + \tfrac{1}{a}) \geq 4\end{aligned}$$

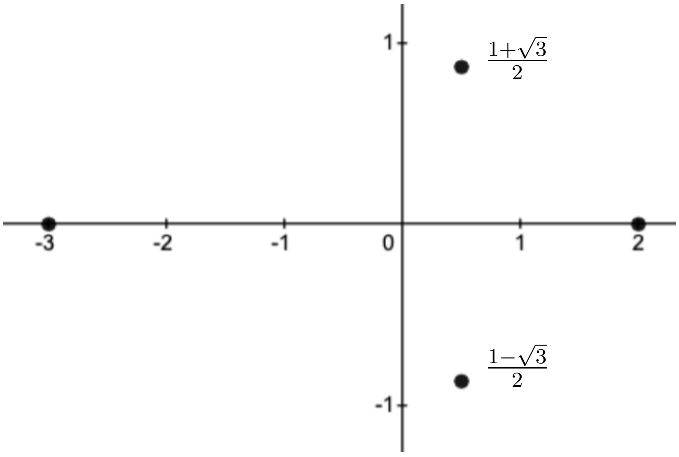
$13bi.\ -6c = -6 \therefore c = 1$ and $(1+3)(1-2)(1^1 + b \times 1 + 1) = -4b - 8 = -4 \therefore b = -1$

$\therefore f(x) = (x+3)(x-2)(x^2 - x + 1)$

Complex roots are $\frac{1 \pm \sqrt{1^2 - 4\times1\times1}}{2} = \frac{1 \pm i\sqrt{3}}{2}$

$13bii.\ \sqrt{(\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}(\cos\tan^{-1}\frac{\sqrt{3}/2}{1/2} \pm i\sin\tan^{-1}\frac{\sqrt{3}/2}{1/2})$
$= 1(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$ or $1(\cos -\frac{\pi}{3} + i\sin -\frac{\pi}{3})$

$13biii.$



$13biv.$ Kite

$13ci.$ $\sqrt{(5-3)^2+(-10+12)^2+(3-4)^2}=3 \therefore (5,-10,3)$ lies on the sphere.

$13cii.$ Line through $(0,0,0)$ and $(3,-12,4)$ is $\frac{x-3}{-3}=\frac{y+12}{12}=\frac{z-4}{-4}$

Sphere is $(x-3)^2+(y+12)^2+(z-4)^2=9$

Intersections are such that $(x-3)^2+(-4(x-3))^2+(\frac{4}{3}(x-3))^2=9 \therefore 169(x-3)^2=81$

Hence $x=\frac{39\pm 9}{13}$ and so $y=-4(\frac{39\pm 9}{13}), z=\frac{4}{3}(\frac{39\pm 9}{13})$ respectively which are $(\frac{48}{13},\frac{-192}{13},\frac{64}{13})$, $(\frac{30}{13},\frac{-120}{13},\frac{40}{13})$ and $\sqrt{(\frac{48}{13})^2+(\frac{192}{13})^2+(\frac{64}{13})^2}=16$ and $\sqrt{(\frac{30}{13})^2+(\frac{120}{13})^2+(\frac{40}{13})^2}=10$ so the point furthest from the origin is $(\frac{48}{13},\frac{-192}{13},\frac{64}{13})$.

$13d.$ $3(7)^2-3(7)=126\le 2^7-1=126 \therefore$ it is true for $n=7$.

If it is true for $n=k$ then $3k^2-3k\le 2^k-1$

$$\begin{aligned}\therefore 3(k+1)^2-3(k+1)&=3k^2+6k+3-3k-3\\&=(3k^2-3k)+6k\\&\le 2^k+6k\\&\le 2^k+2^k-1\\&=2^{k+1}-1 \text{ since for } k\ge 7, 6k+1\le 2^k.\end{aligned}$$

Hence by the principle of mathematical induction it is true for all integers $n\ge 7$.

$14ai.$ $z^5-1=(z-1)(z^4+z^3+z^2+z+1)=0$ and $z\ne 1$
$\therefore \frac{z^4+z^3+z^2+z+1}{z^2}=z^2+z+1+z^{-1}+z^{-2}=z^2+2+z^{-2}+z+z^{-1}-1=0$
$\therefore (z+z^{-1})^2+(z+z^{-1})-1=0$

$14aii.$ $\frac{z+z^{-1}}{2} = \frac{\cos\theta + i\sin\theta + \cos(-\theta) + i\sin(-\theta)}{2} = \frac{\cos\theta + i\sin\theta + \cos\theta - i\sin\theta}{2} = \cos\theta$

$14aiii.$ $z^5 = 1 = e^{2k\pi i} \forall k \in \mathbb{Z} \therefore z = e^{\frac{2k\pi i}{5}}, k = 1, 2, 3, 4$
So considering the first of these, $z = e^{\frac{2\pi i}{5}}$ then $\Re(z) = \cos\frac{2\pi}{5}$
Also $\cos\frac{2\pi}{5} = \frac{z+z^{-1}}{2} = \frac{1}{2} \cdot \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} = \frac{-1 \pm \sqrt{5}}{4}$ and since $\cos\frac{2\pi}{5} > 0, \cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$.

$14aiv.$ Likewise for $z_3 = e^{\frac{4\pi i}{5}}, \Re(z_3) = \cos\frac{4\pi}{5} = \frac{z_3 + z_3^{-1}}{2} = \frac{-1-\sqrt{5}}{4}$

Now by the cosine rule, $|z_2 - z_1|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos\frac{2\pi}{5} = 1 + 1 - 2(1)(1)\frac{\sqrt{5}-1}{4}$

$\therefore |z_2 - z_1|^2 = \frac{5-\sqrt{5}}{2}$

Also, $|z_3 - z_1|^2 = |z_1|^2 + |z_3|^2 - 2|z_1||z_3|\cos\frac{4\pi}{5} = 1 + 1 - 2(1)(1)\frac{-\sqrt{5}-1}{4} = \frac{5+\sqrt{5}}{2}$

$\therefore \frac{|z_3-z_1|^2}{|z_2-z_1|^2} = \frac{(5+\sqrt{5})/2}{(5-\sqrt{5})/2} \cdot \frac{5+\sqrt{5}}{5+\sqrt{5}} = \frac{3+\sqrt{5}}{2} = \frac{6+2\sqrt{5}}{4} = \frac{(1+\sqrt{5})^2}{4} \therefore \frac{|z_3-z_1|}{|z_2-z_1|} = \frac{1+\sqrt{5}}{2}$

$14bi.$ $\cos\frac{3\pi}{4}\underset{\sim}{i} + (\cos\frac{3\pi}{4} + \sin\frac{3\pi}{4})\underset{\sim}{j} + \sin\frac{3\pi}{4}\underset{\sim}{k} = -\frac{1}{\sqrt{2}}\underset{\sim}{i} + \frac{1}{\sqrt{2}}\underset{\sim}{k}$

$14bii.$ $\underset{\sim}{r}' = -\frac{\pi}{4}\sin\frac{\pi t}{4}\underset{\sim}{i} + (-\frac{\pi}{4}\sin\frac{\pi t}{4} + \frac{\pi}{4}\cos\frac{\pi t}{4})\underset{\sim}{j} + \frac{\pi}{4}\cos\frac{\pi t}{4}\underset{\sim}{k}$

$\therefore \underset{\sim}{r}'(3) = \frac{\pi}{4}(-\sin\frac{3\pi}{4}\underset{\sim}{i} + (-\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4})\underset{\sim}{j} + \cos\frac{3\pi}{4}\underset{\sim}{k}) = -\frac{\pi\sqrt{2}}{8}(\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k})$

Also $\underset{\sim}{r}(3) = \cos\frac{3\pi}{4}\underset{\sim}{i} + (\cos\frac{3\pi}{4} + \sin\frac{3\pi}{4})\underset{\sim}{j} + \sin\frac{3\pi}{4}\underset{\sim}{k} = \frac{\sqrt{2}}{2}(-\underset{\sim}{i} + \underset{\sim}{k})$

$\therefore$ tangent is $\underset{\sim}{t} = \frac{\sqrt{2}}{2}(-\underset{\sim}{i} + \underset{\sim}{k}) + \lambda(\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k})$

$14c.$ Let $u = 1 + \sqrt{x}$. When $x = 0, u = 1$ and when $x = 9, u = 4$.

$du = \frac{1}{2}x^{-\frac{1}{2}}dx = \frac{dx}{2(u-1)} \therefore dx = 2(u-1)\,du$

$$\begin{aligned}\int_0^9 \frac{1}{\sqrt{1+\sqrt{x}}}\,dx &= \int_1^4 \frac{2(u-1)\,du}{\sqrt{u}} \\ &= \int_1^4 (2u^{\frac{1}{2}} - 2u^{-\frac{1}{2}})\,du \\ &= [\tfrac{4}{3}u^{\frac{3}{2}} - 4u^{\frac{1}{2}}]_1^4 \\ &= \tfrac{32}{3} - 8 - (\tfrac{4}{3} - 4) \\ &= \tfrac{16}{3}\end{aligned}$$

$15a.$ Let $I = \int e^{-x}\cos x\,dx$. Then

$$
\begin{aligned}
I &= \int e^{-x} \tfrac{d}{dx} \sin x \, dx \\
&= e^{-x} \sin x - \int \sin x \tfrac{d}{dx} e^{-x} \, dx \\
&= e^{-x} \sin x + \int e^{-x} \sin x \, dx \\
&= e^{-x} \sin x + \int e^{-x} \tfrac{d}{dx}(-\cos x) \, dx \\
&= e^{-x} \sin x - e^{-x} \cos x - \int -\cos x \tfrac{d}{dx} e^{-x} \, dx \\
&= e^{-x}(\sin x - \cos x) - \int e^{-x} \cos x \, dx \\
&= e^{-x}(\sin x - \cos x) - I
\end{aligned}
$$

$$\therefore 2I = e^{-x}(\sin x - \cos x) + 2C \text{ for a constant } C$$

$$\therefore I = \tfrac{1}{2} e^{-x}(\sin x - \cos x) + C$$

$$
\begin{aligned}
\frac{\int_0^{\frac{\pi}{2}} e^{-x} \cos x \, dx}{\left|\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{-x} \cos x \, dx\right|} &= \frac{[\frac{1}{2} e^{-x}(\sin x - \cos x)]_0^{\frac{\pi}{2}}}{[\frac{1}{2} e^{-x}(\cos x - \sin x)]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}} \\
&= \frac{e^{-\frac{\pi}{2}}(1-0)-(0-1)}{e^{-\frac{3\pi}{2}}(0+1) - e^{-\frac{\pi}{2}}(0-1)} \cdot \frac{e^{\frac{3\pi}{2}}}{e^{\frac{3\pi}{2}}} \\
&= \frac{e^{\pi} + e^{\frac{3\pi}{2}}}{1 + e^{\pi}} \\
&= \frac{e^{\pi}(e^{\frac{\pi}{2}} + 1)}{e^{\pi} + 1}
\end{aligned}
$$

15b. 144km/h$= \frac{144000}{3600}$m/s$=$ 40m/s.

Acceleration up the ramp due to gravity is $-10 \sin 30° = -5\text{ms}^{-2}$

Acceleration up the ramp due to friction is $-0.05v$

Hence $v\frac{dv}{dx} = -0.05v - 5$ and so the distance the car travelled up the ramp before stopping is

$$
\begin{aligned}
\int_{40}^{0} \frac{-v \, dv}{0.05v+5} &= \int_{40}^{0} (-20 + 2000 \cdot \tfrac{0.05}{0.05v+5}) \, dv \\
&= [-20v + 2000 \ln(0.05v + 5)]_{40}^{0} \\
&= 2000 \ln 5 - (-800 + 2000 \ln 7) \\
&= 800 - 2000 \ln 1.4 \\
&\approx 127\text{m}
\end{aligned}
$$

15ci. $2 \sin((n-1)x) \sin x = \cos((n-1)x - x) - \cos((n-1)x + x) = \cos((n-2)x) - \cos nx$

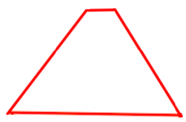
15cii.
$$
\begin{aligned}
I_n - I_{n-2} &= \int \frac{\cos nx - \cos((n-2)x)}{\sin x} \, dx \\
&= \int -2 \sin((n-1)x) \, dx \\
&= \frac{2\cos((n-1)x)}{n-1} + c
\end{aligned}
$$

$15ciii.\ \int_0^{\frac{\pi}{3}} \frac{\cos 2x - \cos 6x}{\sin x}\,dx = [I_2 - I_6]_0^{\frac{\pi}{3}}$

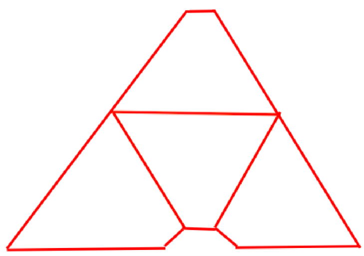
$$= -[I_6 - I_4 + I_4 - I_2]_0^{\frac{\pi}{3}}$$
$$= -[\frac{2\cos 5x}{5} + \frac{2\cos 3x}{3}]_0^{\frac{\pi}{3}}$$
$$= -(\frac{1}{5} - \frac{2}{3} - (\frac{2}{5} + \frac{2}{3}))$$
$$= \frac{23}{15}$$

$16a.$ It is true for $n = 1$ because if T_1 is an equilateral triangle of side length 2 with an equilateral triangle of side length 1 removed at the top it is tiled by $\frac{4^1-1}{3} = 1$ tile in the shape of T_1:

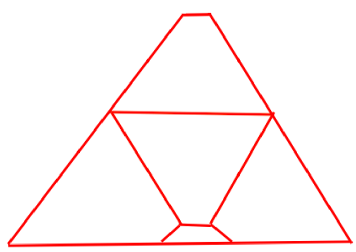
Suppose it is true for $n = k$ then there is a shape T_k which is an equilateral triangle of side length 2^k with an equilateral triangle of side length 1 removed at the top tiled by $\frac{4^k-1}{3}$ tiles in the shape of T_1.



We may take 4 of the T_k shapes and position them like so:



Now we can place T_1 on the base like so thereby making an equilateral triangle of side length 2^{k+1} with an equilateral triangle of side length 1 removed at the top creating a shape T_{k+1} tiled by $4 \cdot \frac{4^k-1}{3} + 1 = \frac{4^{k+1}-1}{3}$ tiles in the shape of T_1 for which the statement is also true for $n = k + 1$:



From this we conclude by the principle of mathematical induction that it is true for all positive integers n. Furthermore the shape T_n which is an equilateral triangle of side length 2^n with an equilateral triangle with side length 1 removed from the top is tiled by $\frac{4^n-1}{3}$ tiles T_1.

Note that this question did not ask for a formula for the number of tiles required. Hence it can be done without reference to the number of tiles.

The proof presented here is really a double induction - proving both the tileability of T_n AND the formula for the number of tiles in T_n.

$16bi$. $(7\sin\frac{\pi t}{32})^2 + (7\cos\frac{\pi t}{32})^2 + (\frac{t}{15})^2 = 49 + \frac{t^2}{225} = 25^2 = 625 \therefore t = 360$ and
$Q = (-\frac{7}{\sqrt{2}}, -\frac{7}{\sqrt{2}}, 24)$

$16bii$. The curve crosses the line when $\frac{\pi t}{32} = 2\pi k - \frac{3\pi}{2}$ for positive integers k.
$\therefore t = 64k - 48$. The greatest such k such that $49 + \frac{t^2}{225} < 625$ is such that
$49 + \frac{(64k-48)^2}{225} < 625 \therefore 64k - 48 < 360 \therefore k < 6.375$.
Hence the largest such $k = \lfloor 6.375 \rfloor = 6$.

$16biii$. Shortest distance from Q to P on the cylinder is $\sqrt{(r\theta)^2 + 24^2}$ where $r = 7$ and θ is the angle between the projection of Q onto the xy-plane and the positive y-axis $= \cos^{-1}\frac{-7/\sqrt{2}}{7} = \frac{3\pi}{4}$. Hence the shortest distance is $\sqrt{\frac{441\pi^2}{16} + 576} \approx 29.1$cm.

$16c$. Going up: $v\frac{dv}{dx} = -10 - 0.01v^2$ stops at height h

$$\begin{aligned}\Rightarrow h &= \int_0^h dx \\ &= \int_{v_0}^0 \frac{-v\,dv}{10+0.01v^2} \\ &= -50\int_{v_o}^0 \frac{2v\,dv}{1000+v^2} \\ &= -50[\ln(1000+v^2)]_{v_0}^0 \\ &= 50\ln(1+0.001v_0^2)\end{aligned}$$

Going down: $v\frac{dv}{dx} = -10 + 0.01v^2$

$$\begin{aligned}h &= -\int_h^0 dx \\ &= -\int_0^{-v_1} \frac{v\,dv}{-10+0.01v^2} \\ &= -50\int_0^{-v_1} \frac{-2v\,dv}{1000-v^2} \\ &= -50[\ln|1000-v^2|]_0^{-v_1} \\ &= 50\ln|1-0.001v_1^2|^{-1}\end{aligned}$$

$\therefore |1-0.001v_1^2|^{-1} = 1 + 0.001v_0^2$ and since $v_1 = \frac{v_0}{7}$, $|1-0.001(\frac{v_0}{7})^2|^{-1} = 1 + 0.001v_0^2$

Now $(1+0.001v_0^2)^2(1-0.001(\frac{v_0}{7})^2)^2 = 1$

$\therefore (v_0^4 - 48000v_0^2 - 98000000)(v^2 - 48000)v_0^2 = 0$ and $v_0 \neq 0$

$\therefore v_0^2 = 48000, \frac{48000+\sqrt{48000^2-4\times1\times(-98000000)}}{2(1)} = 48000, 1000(24+\sqrt{674})$

$\therefore v_0 = 40\sqrt{30}, 10\sqrt{10(24+\sqrt{674})}$

As x is a continuous function of v we have that when going down, $v < 10\sqrt{10}$.

$\frac{40}{7}\sqrt{30} < 10\sqrt{10}$ and so we accept this one.

However $\frac{10}{7}\sqrt{10(24+\sqrt{674})} > 10\sqrt{10}$ and so we reject this one.

Hence $v_0 = 40\sqrt{30} \approx 219.1\text{ms}^{-1}$.