HSC 2023 Mathematics Extension 2 Solutions

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1. C 2. D 3. B 4. C. 5. A 6. B 7. A 8. D 9. D 10. B

Reasoning for Multiple Choice in HSC exams is not required however it is instructive to include them here.

For this reasoning, elimination (eg. it is not A, B nor $C \therefore$ it is D) followed by validation (giving reasons for why it is D when D is the correct answer) is provided for each question.

Efficiency of reasoning: For Q1-9 validation is more efficient. For Q10 elimination is more efficient.

1. Elimination.

$$\begin{aligned}(a+ib)^3 &= a^3+3ia^2b-3ab^2-ib^3\\ &= (a^3-3ab^2)+i(3a^2b+b^3)-2ib^3\\ &\neq (a^3-3ab^2)+i(3a^2b+b^3) \therefore \text{ it is not } A\end{aligned}$$

$$\begin{aligned}(a+ib)^3 &= a^3+3ia^2b-3ab^2-ib^3\\ &= (a^3+3ab^2)+i(3a^2b+b^3)-6ab^2-2ib^3\\ &\neq (a^3+3ab^2)+i(3a^2b+b^3) \therefore \text{ it is not } B\end{aligned}$$

$$\begin{aligned}(a+ib)^3 &= a^3+3ia^2b-3ab^2-ib^3\\ &= (a^3+3ab^2)+i(3a^2b-b^3)-6ab^2\\ &\neq (a^3+3ab^2)+i(3a^2b-b^3) \therefore \text{ it is not } D\end{aligned}$$

It is not A, B nor D hence it is C.

1. Validation

$$\begin{aligned}(a+ib)^3 &= a^3+3ia^2b-3ab^2-ib^3\\ &= (a^3-3ab^2)+i(3a^2b-b^3)\end{aligned}$$

$\therefore$ it is C

2. Elimination

$p =$ an animal is a herbivore, $q =$ it eats meat.

The statement is $p \Rightarrow \neg q$

A is $p \Rightarrow q$ which is not the converse $\therefore$ it is not A

B is $\neg p \Rightarrow q$ which is not the converse $\therefore$ it is not B

C is $q \Rightarrow \neg p$ which is not the converse $\therefore$ it is not C

It is not A, B nor C $\therefore$ it is D

2. Validation

D is $\neg q \Rightarrow p$ which is the converse $\therefore$ it is D.

3. Elimination

$z = e^{-\frac{2\pi i}{3}}$

$-z = e^{\pi i} \cdot e^{-\frac{2\pi i}{3}} = e^{\frac{\pi i}{3}} \neq \bar{z} \therefore$ it is not A

$-z^3 = e^{\pi i} \cdot e^{-2\pi i} = -1 \neq \bar{z} \therefore$ it is not C

$z^4 = z(z^3) = z(1) = z \neq \bar{z} \therefore$ it is not D

It is neither A, C nor D $\therefore$ it is B

3. Validation

$z^2 = e^{-\frac{4\pi i}{3} + 2\pi i} = e^{\frac{2\pi i}{3}} = \bar{z} \therefore$ it is B

4. Elimination.

A is "Whichever number x greater than 1 you pick, it is possible to find a positive number r such that $\frac{\ln x}{x^3} < r$"

$\therefore$ it is not A

B is "It is possible to find a number x greater than 1 such that for whichever positive number r you pick, $\frac{\ln x}{x^3} < r$"

$\therefore$ it is not B

D is "It is possible to find a positive number r such that for whichever number x greater than 1 you pick, $\frac{\ln x}{x^3} < r$"

$\therefore$ it is not D

It is not A, B nor D $\therefore$ it is C

4. Validation

C is "Whichever positive number r you pick, it is possible to find a number x greater than 1 such that $\frac{\ln x}{x^3} < r$"

$\therefore$ it is C.

5. Elimination

$\ell_2 = \begin{pmatrix} 3 \\ -10 \\ 1 \end{pmatrix} - \mu \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$ and so ℓ_1 and ℓ_2 are parallel and so it is not B nor D

With $\lambda = -4$, $\begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} - 4 \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -10 \\ 1 \end{pmatrix}$ and so $(3, -10, 1)$ is on ℓ_1 and so it is not correct to say they do not intersect, hence it is not C

It is not B, C nor D $\therefore$ it is A

5. Validation.

$(3, -10, 1)$ is on ℓ_1 and they are parallel and so they are the same line. Hence it is A

6. Elimination.

For A, $x = \frac{1}{2}(\cos 2t + 1) - \sin 2t = \frac{1}{2}(\cos 2t - 2\sin 2t) + \frac{1}{2}$
$\dot{x} = 2\cos t(-\sin t) - 2\cos 2t = -\sin 2t - 2\cos 2t$
$\ddot{x} = -2\cos 2t + 4\sin 2t = -4(\frac{1}{2}(\cos 2t - 2\sin 2t)) = -4(x - \frac{1}{2})$ which is simple harmonic so it is not A

For C, $\dot{x} = 6\cos 3t + 12\sin 3t$

$\ddot{x} = -18\sin 3t + 36\cos 3t = -9(2\sin 3t - 4\cos 3t) = -9(x-5)$ which is simple harmonic and so it is not C

For D, $\dot{x} = -8\sin(2t + \frac{\pi}{2}) + 10\cos(2t - \frac{\pi}{4})$

$\ddot{x} = -16\cos(2t + \frac{\pi}{2}) - 20\sin(2t - \frac{\pi}{4}) = -4(4\cos(2t + \frac{\pi}{2}) + 5\sin(2t - \frac{\pi}{4}))) = -4x$ which is simple harmonic and hence it is not D.

It is not A, C not D $\therefore$ it is B

6. Validation

For B, $\dot{x} = 4\cos 4t - 8\sin 2t$

$\ddot{x} = -16\sin 4t - 16\cos 2t$ and $\dddot{x} = -64\cos 4t + 32\sin 2t$

If it were simple harmonic we would have $\ddot{x} = -n^2(x - c)$ for some positive number n and a real number c in which case $\frac{d\ddot{x}}{dx}$ would be the negative constant $-n^2$.

But $\frac{d\ddot{x}}{dx} = \frac{\dddot{x}}{\dot{x}} = \frac{-64\cos 4t+32\sin 2t}{4\cos 4t-8\sin 2t}$ which is not constant and so it is not simple harmonic in which case the answer is B

To see that it not constant we can substitute values for t and get different results, eg. $\frac{d\ddot{x}}{dx}|_{t=0} = -16$ but $\frac{d\ddot{x}}{dx}|_{t=-\frac{\pi}{4}} = 8$

7. Elimination

$\text{Arg}(-i) = \text{Arg}(-2i) = -\frac{\pi}{2}$ and so $\text{Arg}(-i) + \text{Arg}(-2i) = -\pi$

But $\text{Arg}(i \cdot 2i) = \text{Arg}(-2) = \pi \neq -\pi$ so it is not B

$e^{0i} = e^{2\pi i} = 1$ but $0 \neq 2\pi$ so it is not C

$-1 - i = \sqrt{2}e^{-\frac{3\pi i}{4}}$ and $\arctan(\frac{-1}{-1}) = \frac{\pi}{4} \neq -\frac{3\pi}{4}$ so it is not D

It is not B, C nor D hence it is A

7. Validation

If $\theta = \arg(x + iy)$ and $x \neq 0$ then $\tan\theta = \frac{y}{x}$ and $x + iy = re^{i\theta}$ where $r = \sqrt{x^2 + y^2}$

Hence it is A.

8. Elimination

For A, shaded region includes 0 but $|0 - i| = 1 < 2 = 2|0 - 1|$ so it is not A

For B, shaded region included 1 but $|1 - i| = \sqrt{2} > 0 = 2|1 - 1|$ so it is not B

For C, shaded region includes 1 but $|1 - 1| = 0 < 2\sqrt{2} = 2|1 - i|$ so it is not C

It is not A, B nor C and hence it is D

8. Validation

For D, $(x-1)^2 + y^2 < 4(x^2 + (y-1)^2)$

$\therefore x^2 - 2x + 1 + y^2 < 4x^2 + 4y^2 - 8y + 4$

$\therefore 3x^2 + 2x + 3y^2 - 8y + 3 > 0$

$\therefore x^2 + \frac{2}{3}x + \frac{1}{9} + y^2 - \frac{8}{3}y + \frac{16}{9} > \frac{17}{9} - 1$

$\therefore (x + \frac{1}{3})^2 + (y - \frac{4}{3})^2 > \frac{8}{9}$

which is outside a circle with centre $(-\frac{1}{3}, \frac{4}{3})$ and radius $\frac{2\sqrt{2}}{3}$ which concurs with the shaded region in the question and so it is D.

9. Elimination

$\mathbf{r}$ and $\text{Proj}_{\mathbf{r}}\mathbf{v}$ are in the same direction and so $\mathbf{r} \cdot \text{Proj}_{\mathbf{r}}\mathbf{v} \geq 0 \therefore \mathbf{r} \cdot \mathbf{v} \geq 0$ and so it is not A nor B

$\text{Proj}_{\mathbf{v}}\mathbf{a}$ and $\mathbf{v}$ are in opposite directions and so $\mathbf{v} \cdot \text{Proj}_{\mathbf{v}}\mathbf{a} \leq 0 \therefore \mathbf{a} \cdot \mathbf{v} \leq 0$ and so it is not C

It is not A, B nor C hence it is D

9. Validation

As from above we see $\mathbf{r} \cdot \mathbf{v} \geq 0$ and $\mathbf{a} \cdot \mathbf{v} \leq 0$ and so it is D.

10. Elimination

For A, if $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are unit vectors and $\underset{\sim}{a} \cdot \underset{\sim}{b} = 1$ then $\underset{\sim}{a} = \underset{\sim}{b}$ and then $\underset{\sim}{b} \cdot \underset{\sim}{c}$ and $\underset{\sim}{c} \cdot \underset{\sim}{a}$ should be the same but they are 2 and 3 respectively

If $\underset{\sim}{a}$ and $\underset{\sim}{c}$ are unit vectors and the angle between them is α then $\underset{\sim}{c} \cdot \underset{\sim}{a} = \cos\alpha = 3$ but $-1 \leq \cos\alpha \leq 1$

Likewise if $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are unit vectors and the angle between them is β then $\underset{\sim}{b} \cdot \underset{\sim}{c} = \cos\beta = 2$ but $-1 \leq \cos\beta \leq 1$

Hence it can't be A

$\underset{\sim}{0} \cdot \underset{\sim}{b} = 0 \neq 1$ hence it can't be C

For D, consider $\underset{\sim}{a} \cdot (r\underset{\sim}{a} + s\underset{\sim}{b} + t\underset{\sim}{c}) = r|\underset{\sim}{a}|^2 + s\underset{\sim}{a} \cdot \underset{\sim}{b} + t\underset{\sim}{c} \cdot \underset{\sim}{a} = r|\underset{\sim}{a}|^2 + s + 3t > 0$. Now if the proposition were true $\underset{\sim}{a} \cdot (r\underset{\sim}{a} + s\underset{\sim}{b} + t\underset{\sim}{c}) = \underset{\sim}{a} \cdot \underset{\sim}{0} = 0$ - contradiction, hence it isn't true and it can't be D

It is not A, C nor D hence it is B

10. Validation

Suppose $A = (2, 0, 0), B = (\frac{1}{2}, \frac{\sqrt{15}}{2}, 0)$ and $C = (\frac{3}{2}, \frac{\sqrt{15}}{6}, \frac{2\sqrt{3}}{3})$

Then A, B, C lie on the sphere $x^2 + y^2 + z^2 = 4$

Check $A : 2^2 + 0^2 + 0^2 = 4$

Check $B : (\frac{1}{2})^2 + (\frac{\sqrt{15}}{2})^2 + 0^2 = 4$ and

Check $C : (\frac{3}{2})^2 + (\frac{\sqrt{15}}{6})^2 + (\frac{2\sqrt{3}}{3})^2 = 4$

Now checking the dot products

$\underset{\sim}{a} \cdot \underset{\sim}{b} = 2 \times \frac{1}{2} + 0 \times \frac{\sqrt{15}}{2} + 0 \times 0 = 1$

$\underset{\sim}{b} \cdot \underset{\sim}{c} = \frac{1}{2} \times \frac{3}{2} + \frac{\sqrt{15}}{2} \times \frac{\sqrt{15}}{6} + 0 \times \frac{2\sqrt{3}}{3} = 2$

$\underset{\sim}{c} \cdot \underset{\sim}{a} = \frac{3}{2} \times 2 + \frac{\sqrt{15}}{6} \times 0 + \frac{2\sqrt{3}}{3} \times 0 = 3$

Hence the answer is B

11a. $z = \frac{3 \pm \sqrt{3^2 - 4 \times 1 \times 4}}{2} = \frac{3 \pm i\sqrt{7}}{2}$

11b. $\cos^{-1} \frac{1 \times -1 + 2 \times 4 - 3 \times 2}{\sqrt{1^2+2^2+3^2}\sqrt{1^2+4^2+2^2}} = \cos^{-1} \frac{1}{\sqrt{294}} \approx 87°$

11c. If $O = (0, 0, 0)$ and the line is $\mathbf{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB} \quad \forall\lambda \in \mathbb{R}$ then

$$\mathbf{r} = \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 0+3 \\ 2-1 \\ 3-5 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$$

11d. Since $ABCD$ is a parallelogram $\overrightarrow{AB} = \overrightarrow{DC}$

Since $ABEF$ is a parallelogram $\overrightarrow{AB} = \overrightarrow{FE}$

$\therefore \overrightarrow{DC} = \overrightarrow{FE} \therefore \overrightarrow{CD} = \overrightarrow{EF}$ and so $CDFE$ is a parallelogram.

11e. Period$= \frac{2\pi}{\sqrt{9}} = \frac{2\pi}{3}$ and centre is $x = 4$

11f. $\frac{5x-3}{(x+1)(x-3)} \equiv \frac{A}{x+1} + \frac{B}{x-3} \Rightarrow 5x - 3 \equiv A(x-3) + B(x+1) \equiv (A+B)x + B - 3A$

$\therefore A + B = 5$ and $B - 3A = -3 \therefore A + B - (B - 3A) = 4A = 5 + 3 = 8$ and so $A = 2$ and $B = 5 - 2 = 3$ whereupon $\frac{5x-3}{(x+1)(x-3)} \equiv \frac{2}{x+1} + \frac{3}{x-3}$

$$\begin{aligned}\int_0^2 \frac{5x-3}{(x+1)(x-3)} &= \int_0^2 \left(\frac{2}{x+1} + \frac{3}{x-3}\right) dx \\ &= [2\ln|x+1| + 3\ln|x-3|]_0^2 \\ &= 2\ln 3 + 3\ln 1 - (2\ln 1 + 3\ln 3) \\ &= -\ln 3\end{aligned}$$

12a. If $\sqrt{23} \in \mathbb{Q}$ then $\exists$ coprime $p, q : \frac{p}{q} = \sqrt{23} \therefore \frac{p^2}{q^2} = 23 \therefore p^2 = 23q^2 \therefore 23|p^2 \therefore 23|p$

Hence $\exists x \in \mathbb{Z} : p = 23x \therefore (23x)^2 = 23q^2$ and so $23x^2 = q^2 \therefore 23|q^2$ and so $23|q$ contradicting coprimality of p, q

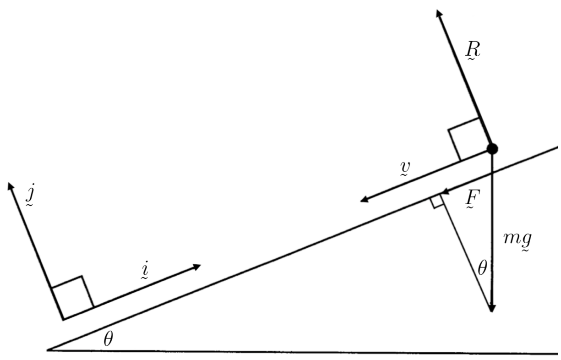
Hence $\sqrt{23}$ is irrational

12b. $\forall x, y \in \mathbb{R} : x^2 + y^2 \neq 0$ we have

$$\begin{aligned} (x-y)^2 &= x^2 - 2xy + y^2 \\ &\geq 0 \\ \iff \quad x^2 + 2xy + y^2 &\leq 2x^2 + 2y^2 \\ \iff \quad \frac{(x+y)^2}{x^2+y^2} &\leq 2 \end{aligned}$$

Hence $\frac{(x+y)^2}{x^2+y^2} \leq 2$

12ci.



From the diagram $\sin\theta = \frac{|\underset{\sim}{F}|}{mg}$ and as $\underset{\sim}{F}$ is in the opposite direction to $\underset{\sim}{i}$, $\underset{\sim}{F} = -(mg\sin\theta)\underset{\sim}{i}$

12cii. With acceleration $\underset{\sim}{a}(t), \underset{\sim}{F} = m\underset{\sim}{a}(t) = -mg\sin\theta\underset{\sim}{i}$ and so we have

$$\underset{\sim}{a}(t) = -g\sin\theta\underset{\sim}{i}$$
$$\begin{aligned}\underset{\sim}{v}(t) &= \int_0^t -g\sin\theta\underset{\sim}{i}\,dT\\ &= [-gT\sin\theta\underset{\sim}{i}]_0^t\\ &= -gt\sin\theta\underset{\sim}{i}+\underset{\sim}{0}\\ &= -gt\sin\theta\underset{\sim}{i}\end{aligned}$$

12d. If $z^3 = 2-2i = \sqrt{2^2+2^2}e^{-i\tan^{-1}\frac{2}{2}} = 2\sqrt{2}e^{-\frac{\pi i}{4}+2\pi ki} = 2\sqrt{2}e^{\frac{(8k-1)\pi i}{4}} \quad \forall k\in\mathbb{Z}$

then $z = \sqrt{2}e^{\frac{(8k-1)\pi i}{12}}$ for $k = 0, \pm 1$

$\Rightarrow z = \sqrt{2}e^{-\frac{\pi i}{12}}, \sqrt{2}e^{\frac{7\pi i}{12}}, \sqrt{2}e^{-\frac{3\pi i}{4}}$

12ei. Coefficients are all real and so non-real zeros occur as conjugate pairs and since $2+i$ is a zero so too is $\overline{2+i} = 2-i$.

12eii. If remaining zeros are α,β then $\alpha+\beta+2+i+2-i = \alpha+\beta+4 = -\frac{-3}{1} = 3 \therefore$ $\alpha+\beta = -1$. Also $\alpha\beta(2+i)(2-i) = 5\alpha\beta = \frac{-30}{1} = -30$ and so $\alpha\beta = -6$

$\beta = -1-\alpha$ so $\alpha(-1-\alpha) = -\alpha-\alpha^2 = -6$ and so $\alpha^2+\alpha-6 = (\alpha+3)(\alpha-2) = 0$ so w.l.o.g., $\alpha = 2, \beta = -3$

13a. Let $x = -2+3\sin\theta$. Then $dx = 3\cos\theta\,d\theta,\ \theta = \sin^{-1}\frac{x+2}{3}$ and $1-x = 3-3\sin\theta$.

Also $\cos\theta = \sqrt{1-(\frac{x+2}{3})^2} = \frac{\sqrt{5-4x-x^2}}{3}$. Hence

$$\begin{aligned}\int\frac{1-x}{\sqrt{5-4x-x^2}}\,dx &= \int\frac{3-3\sin\theta}{3\cos\theta}\cdot 3\cos\theta\,d\theta\\ &= \int(3-3\sin\theta)\,d\theta\\ &= 3\theta+3\cos\theta+C\\ &= 3\sin^{-1}\frac{x+2}{3}+\sqrt{5-4x-x^2}+C\end{aligned}$$

13bi. $\frac{d}{dk}(k^2-2k-3) = 2k-2 > 0$ when $k>1$ and so is increasing for $k>1$. Also $k^2-2k-3 = (k-3)(k+1) = 0 \Rightarrow k = -1, 3 \therefore k^2-2k-3 \geq 0$ for $k\geq 3$.

13bii. The statement is true for $n=3$ since $2^3 = 8 \geq 7 = 3^2-2$

If the statement is true for $n=k$ then $2^k \geq k^2-2$ and so

$$\begin{aligned}2^{k+1}-(k+1)^2+2 &= 2(2^k)-k^2-2k+1\\ &\geq 2(k^2-2)-k^2-2k+1\\ &= k^2-2k-3\\ &\geq 0 \text{ for } k\geq 3 \text{ from 13bi}\end{aligned}$$

and so $2^{k+1} \geq (k+1)^2 - 2$ which means the statement is true for $n = k+1$.

Therefore by the principle of mathematical induction the statement is true for all positive integers $n \geq 3$.

13ci. $\mathbf{v}(0) = \begin{pmatrix} 40\cos 30^\circ \\ 40\sin 30^\circ \end{pmatrix} = \begin{pmatrix} 20\sqrt{3} \\ 20 \end{pmatrix}$

13cii. If $\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ then $\ddot{x} = -4\dot{x}$ and $\ddot{y} = -10 - 4\dot{y}$

$\int \frac{d\dot{x}}{\dot{x}} = \int -4\,dt \therefore \ln \dot{x} = -4t + C_1$ and $t = 0 \Rightarrow \dot{x} = 20\sqrt{3} \therefore C_1 = \ln(20\sqrt{3})$.

Now $\ln \dot{x} = -4t + \ln(20\sqrt{3})$ and so $\dot{x} = e^{-4t+\ln(20\sqrt{3})} = 20\sqrt{3}e^{-4t}$

Likewise $\int \frac{d\dot{y}}{\frac{5}{2}+\dot{y}} = \int -4\,dt \therefore \ln(\frac{5}{2}+\dot{y}) = -4t+C_2$ and $t = 0 \Rightarrow \dot{y} = 20$ and so $C_2 = \ln\frac{45}{2}$

$\frac{5}{2} + \dot{y} = e^{-4t+\ln\frac{45}{2}} = \frac{45}{2}e^{-4t}$ and so $\dot{y} = \frac{45}{2}e^{-4t} - \frac{5}{2}$

Hence $\mathbf{v}(t) = \begin{pmatrix} 20\sqrt{3}e^{-4t} \\ \frac{45}{2}e^{-4t} - \frac{5}{2} \end{pmatrix}$

13ciii. $x = \int 20\sqrt{3}e^{-4t}\,dt = -5\sqrt{3}e^{-4t} + C_3$ and $t = 0 \Rightarrow x = 0$ and so $C_3 = 5\sqrt{3}$ whereupon $x = 5\sqrt{3}(1 - e^{-4t})$

Likewise $y = \int(\frac{45}{2}e^{-4t} - \frac{5}{2})\,dt = -\frac{45}{8}e^{-4t} - \frac{5t}{2} + C_4$ and $t = 0 \Rightarrow y = 0$ and so

$C_4 = \frac{45}{8}$ whereupon $y = \frac{45}{8}(1 - e^{-4t}) - \frac{5}{2}t$ and so

$\mathbf{r}(t) = \begin{pmatrix} 5\sqrt{3}(1 - e^{-4t}) \\ \frac{45}{8}(1 - e^{-4t}) - \frac{5}{2}t \end{pmatrix}$

13civ. $\frac{45}{8}(1 - e^{-4t}) - \frac{5}{2}t = 0 \Rightarrow 1 - e^{-4t} = \frac{4t}{9}$ and from the graph $t \approx 2.25$ and so range $\approx x(2.25) = 5\sqrt{3}(1 - e^{-4\times 2.25}) \approx 8.7$m

14ai
$$\begin{aligned} |z+w|^2 &= \left|\frac{\sqrt{3}}{2} + \frac{i}{2} - \frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}\right| \\ &= \left(\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\right)^2 + \left(\frac{1}{2} + \frac{\sqrt{2}}{2}\right)^2 \\ &= \frac{3}{4} - \frac{\sqrt{6}}{2} + \frac{1}{2} + \frac{1}{4} + \frac{\sqrt{2}}{2} + \frac{1}{2} \\ &= \frac{4 - \sqrt{6} + \sqrt{2}}{2} \end{aligned}$$

14aii. As $OACB$ is a rhombus, OC bisects $\angle AOB$. Hence $\angle AOC = \frac{1}{2}(\frac{3\pi}{4} - \frac{\pi}{6}) = \frac{7\pi}{24}$

14aiii.Since diagonals AB and OC of the rhombus bisect each other at right angles,

$$\begin{aligned}
\cos\frac{7\pi}{24} &= \frac{\frac{1}{2}OC}{OA} \\
&= \frac{\frac{1}{2}|z+w|}{|z|} \\
&= \frac{1}{2}\sqrt{\frac{4-\sqrt{6}+\sqrt{2}}{2}} \\
&= \frac{1}{2}\sqrt{\frac{8-2\sqrt{6}+2\sqrt{2}}{4}} \\
&= \frac{\sqrt{8-2\sqrt{6}+2\sqrt{2}}}{2}
\end{aligned}$$

14b. If n is the angular frequency then $\frac{2\pi}{n} = 8\pi \therefore n = \frac{1}{4}$ and so the first time they collide after $t = 2\pi$ s is such that $4\cos\frac{t}{4} = 4\cos\frac{t-2\pi}{4} = 4\sin\frac{t}{4} \therefore \tan\frac{t}{4} = 1$ so $\frac{t}{4} = \frac{5\pi}{4}$

Now $t = 5\pi$ and $4\cos\frac{5\pi}{4} = -2\sqrt{2}$. Hence they first collide after 5π s at $2\sqrt{2}$ m to the left of the origin.

14ci. If acceleration is a then when going up, $Ma = -Mg - kMv^2$
so $a = v\frac{dv}{dx} = -g - kv^2$ and so

$$\begin{aligned}
\int_{v_0}^{0} \frac{v\,dv}{g+kv^2} = \frac{1}{2k}\int_{v_0}^{0}\frac{2kv\,dv}{g+kv^2} &= \int_0^H -dx \\
\left[\frac{1}{2k}\ln(g+kv^2)\right]_{v_0}^{0} &= [-x]_0^H \\
\frac{1}{2k}\ln g - \frac{1}{2k}\ln(g+kv_0^2) &= -H+0 \\
\therefore H &= \frac{1}{2k}\ln(g+kv_0^2) - \frac{1}{2k}\ln g \\
&= \frac{1}{2k}\ln\left(\frac{g+kv_0^2}{g}\right)
\end{aligned}$$

14cii. When going down, $Ma = Mg - kMv^2$
so $a = v\frac{dv}{dx} = g - kv^2$ and so

$$\begin{aligned}
\int_0^{v_1} \frac{v\,dv}{g-kv^2} = -\frac{1}{2k}\int_0^{v_1} \frac{-2kv\,dv}{g-kv^2} &= \int_0^H dx \\
\left[-\frac{1}{2k}\ln(g-kv^2)\right]_0^{v_1} &= [x]_0^H \\
-\frac{1}{2k}\ln(g-kv_1^2) + \frac{1}{2k}\ln g &= H - 0 \\
\frac{1}{2k}\ln\left(\frac{g}{g-kv_1^2}\right) &= \frac{1}{2k}\ln\left(\frac{g+kv_0^2}{g}\right) \\
(g-kv_1^2)(g+kv_0^2) &= g^2 \\
g^2 - gkv_1^2 + gkv_0^2 - k^2v_0^2v_1^2 &= g^2 \\
\therefore g(v_0^2 - v_1^2) &= kv_0^2v_1^2
\end{aligned}$$

15ai.

$$\begin{aligned}
J_n &= \int_0^{\frac{\pi}{2}} \sin^n\theta\,d\theta \\
&= \int_0^{\frac{\pi}{2}} \sin\theta\cdot\sin^{n-1}\theta\,d\theta \\
&= \int_0^{\frac{\pi}{2}} \left(\frac{d}{d\theta}(-\cos\theta)\right)\cdot\sin^{n-1}\theta\,d\theta \\
&= [-\cos\theta\sin^{n-1}\theta]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\cos\theta\cdot\frac{d}{d\theta}\sin^{n-1}\theta\,d\theta \\
&= \int_0^{\frac{\pi}{2}} (n-1)\cos^2\theta\sin^{n-2}\theta\,d\theta \\
&= (n-1)\int_0^{\frac{\pi}{2}} (1-\sin^2\theta)\sin^{n-2}\theta\,d\theta \\
&= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta\,d\theta - (n-1)\int_0^{\frac{\pi}{2}} \sin^n\theta\,d\theta \\
&= (n-1)J_{n-2} - (n-1)J_n \\
J_n + (n-1)J_n &= (n-1)J_{n-2} \\
nJ_n &= (n-1)J_{n-2} \\
J_n &= \frac{n-1}{n}J_n \text{ for all integers } n \geq 2
\end{aligned}$$

15aii. $I_n = \int_0^1 x^n(1-x)^n\,dx$ where n is a positive integer and $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\,d\theta, 1-x = \cos^2\theta$, when $x = 0, \theta = 0$ and when $x = 1, \theta = \frac{\pi}{2}$ and now

$$
\begin{aligned}
I_n &= \int_0^{\frac{\pi}{2}} \sin^{2n}\theta\cos^{2n}\theta \cdot 2\sin\theta\cos\theta\, d\theta \\
&= \int_0^{\frac{\pi}{2}} 2\sin^{2n+1}\theta\cos^{2n+1}\theta\, d\theta \\
&= \frac{1}{2^{2n+1}} \int_0^{\frac{\pi}{2}} (2\sin\theta\cos\theta)^{2n+1} \cdot 2\, d\theta \\
&= \frac{1}{2^{2n+1}} \int_0^{\pi} \sin^{2n+1} 2\theta\, d(2\theta) \\
&= \frac{1}{2^{2n+1}} \int_0^{\pi} \sin^{2n+1}\theta\, d\theta \\
&= \frac{1}{2^{2n}} \int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta\, d\theta \text{ by symmetry}
\end{aligned}
$$

15aii (alternative solution).

Where B is the Beta function, using the identity $\int_0^{\frac{\pi}{2}} \sin^{2a-1}\theta\cos^{2b-1}\theta\, d\theta = \frac{1}{2}B(a,b)$ with $a = b = n+1$ we have

$$
\begin{aligned}
I_n &= B(n+1, n+1) \\
&= 2\int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta\cos^{2n+1}\theta\, d\theta \\
&= \frac{2}{2^{2n+1}} \int_0^{\frac{\pi}{2}} (2\sin\theta\cos\theta)^{2n+1}\, d\theta \\
&= \frac{1}{2^{2n}} \int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta\, d\theta
\end{aligned}
$$

15aiii. From 15ai and 15aii,

$$
\begin{aligned}
I_n &= \frac{1}{2^{2n}} J_{2n+1} \\
&= \frac{1}{2^{2n}} \cdot \frac{(2n+1)-1}{2n+1} J_{(2n+1)-2} \\
&= \frac{1}{2^{2n-1}} \cdot \frac{n}{2n+1} J_{2n-1} \\
&= \frac{1}{2^{2n-1}} \cdot \frac{n}{2n+1} J_{2(n-1)+1} \\
&= \frac{1}{2^{2n-1}} \cdot \frac{n}{2n+1} \cdot 2^{2(n-1)} I_{n-1} \\
&= \frac{n}{4n+2} I_{n-1} \text{ for all integers } n \geq 1
\end{aligned}
$$

15aiii. (alternative solution) Since $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ where B is the Beta function and Γ is the Gamma function and $\Gamma(x+1) = x!$ for positive integers x we have that for positive integers a, b, $B(a,b) = \frac{(a-1)!(b-1)!}{(a+b-1)!}$ and so if $a = b = n+1$ then

$$
\begin{aligned}
I_n &= B(n+1, n+1) \\
&= \frac{(n!)^2}{(2n+1)!} \\
&= \frac{((n-1)!n)^2}{(2n+1)2n(2n-1)!} \\
&= \frac{n}{4n+2} \cdot \frac{((n-1)!)^2}{(2n-1)!} \\
&= \frac{n}{4n+2} B(n,n) \\
&= \frac{n}{4n+2} I_{n-1} \text{ for all integers } n \geq 1
\end{aligned}
$$

15bi. $\overrightarrow{LP} = \overrightarrow{LA} + \overrightarrow{AD} + \overrightarrow{DP}$

$$
\begin{aligned}
&= -\frac{1}{2}\underset{\sim}{b} + \underset{\sim}{d} + \frac{1}{2}(\underset{\sim}{c} - \underset{\sim}{d}) \\
&= \frac{1}{2}(-\underset{\sim}{b} + \underset{\sim}{c} + \underset{\sim}{d})
\end{aligned}
$$

15bii. $|\overrightarrow{AB}|^2 + |\overrightarrow{AC}|^2 + |\overrightarrow{AD}|^2 + |\overrightarrow{BC}|^2 + |\overrightarrow{BD}|^2 + |\overrightarrow{CD}|^2 - 4(|\overrightarrow{LP}|^2 + |\overrightarrow{MQ}|^2 + |\overrightarrow{NR}|^2)$

$= |\underset{\sim}{b}|^2 + |\underset{\sim}{c}|^2 + |\underset{\sim}{d}|^2 + |\underset{\sim}{c} - \underset{\sim}{b}|^2 + |\underset{\sim}{d} - \underset{\sim}{b}|^2 + |\underset{\sim}{d} - \underset{\sim}{c}|^2 - 4(\frac{1}{4}|-\underset{\sim}{b} + \underset{\sim}{c} + \underset{\sim}{d}|^2 + \frac{1}{4}|\underset{\sim}{b} - \underset{\sim}{c} + \underset{\sim}{d}|^2 + \frac{1}{4}|\underset{\sim}{b} + \underset{\sim}{c} - \underset{\sim}{d}|^2)$

$= |\underset{\sim}{b}|^2 + |\underset{\sim}{c}|^2 + |\underset{\sim}{d}|^2 + (\underset{\sim}{c} - \underset{\sim}{b}) \cdot (\underset{\sim}{c} - \underset{\sim}{b}) + (\underset{\sim}{d} - \underset{\sim}{b}) \cdot (\underset{\sim}{d} - \underset{\sim}{b}) + (\underset{\sim}{d} - \underset{\sim}{c}) \cdot (\underset{\sim}{d} - \underset{\sim}{c})$

$-(-\underset{\sim}{b} + \underset{\sim}{c} + \underset{\sim}{d}) \cdot (-\underset{\sim}{b} + \underset{\sim}{c} + \underset{\sim}{d}) - (\underset{\sim}{b} - \underset{\sim}{c} + \underset{\sim}{d}) \cdot (\underset{\sim}{b} - \underset{\sim}{c} + \underset{\sim}{d}) - (\underset{\sim}{b} + \underset{\sim}{c} - \underset{\sim}{d}) \cdot (\underset{\sim}{b} + \underset{\sim}{c} - \underset{\sim}{d})$

$= |\underset{\sim}{b}|^2 + |\underset{\sim}{c}|^2 + |\underset{\sim}{d}|^2 + 2|\underset{\sim}{b}|^2 + 2|\underset{\sim}{c}|^2 + 2|\underset{\sim}{d}|^2 - 2\underset{\sim}{b} \cdot \underset{\sim}{c} - 2\underset{\sim}{b} \cdot \underset{\sim}{d} - 2\underset{\sim}{c} \cdot \underset{\sim}{d}$

$-3|\underset{\sim}{b}|^2 - 3|\underset{\sim}{c}|^2 - 3|\underset{\sim}{d}|^2 + 2\underset{\sim}{b} \cdot \underset{\sim}{c} + 2\underset{\sim}{b} \cdot \underset{\sim}{d} - 2\underset{\sim}{c} \cdot \underset{\sim}{d} + 2\underset{\sim}{b} \cdot \underset{\sim}{c} - 2\underset{\sim}{b} \cdot \underset{\sim}{d} + 2\underset{\sim}{c} \cdot \underset{\sim}{d} - 2\underset{\sim}{b} \cdot \underset{\sim}{c} + 2\underset{\sim}{b} \cdot \underset{\sim}{d} + 2\underset{\sim}{c} \cdot \underset{\sim}{d}$

$= 0 \therefore |\overrightarrow{AB}|^2 + |\overrightarrow{AC}|^2 + |\overrightarrow{AD}|^2 + |\overrightarrow{BC}|^2 + |\overrightarrow{BD}|^2 + |\overrightarrow{CD}|^2 = 4(|\overrightarrow{LP}|^2 + |\overrightarrow{MQ}|^2 + |\overrightarrow{NR}|^2)$

15bii. (Alternative solution 1)

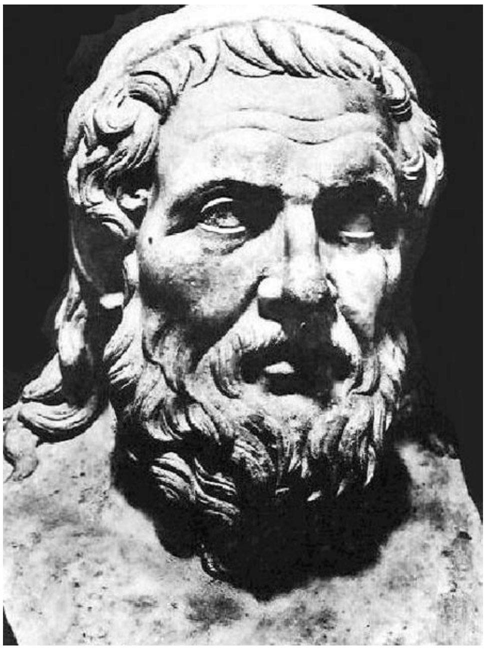
Nathan Altshiller-Court

The sum of the squares of two pairs of opposite edges of a tetrahedron is equal to the sum of the squares of the remaining two opposite edges increased by four times the square of the bimedian relative to these last edges.

Altshiller-Court, N. Modern Pure Solid Geometry. New York: The Macmillan Company 1935, page 56.

As a corollary, the sum of the squares of the edges of a tetrahedron is equal to four times the sum of the squares of its bimedians.

15bii. (Alternative solution 2)



Apollonius of Perga

By Apollonius' Theorem in $\triangle BMD, 2MQ^2 = BM^2 + DM^2 - 2BQ^2$

in $\triangle ABC, 2BM^2 = AB^2 + BC^2 - 2AM^2$

and in $\triangle ADC, 2DM^2 = AD^2 + CD^2 - 2AM^2$ and as $BQ = \frac{1}{2}BD$ and $AM = \frac{1}{2}AC$ we now have

$$\begin{aligned}4MQ^2 &= 2BM^2 + 2DM^2 - BD^2\\ &= AB^2 + BC^2 - \tfrac{1}{2}AC^2 + AD^2 + CD^2 - \tfrac{1}{2}AC^2 - BD^2\\ &= AB^2 + BC^2 + CD^2 + AD^2 - AC^2 - BD^2\end{aligned}$$

Likewise $4LP^2 = AC^2 + BC^2 + BD^2 + AD^2 - AB^2 - CD^2$
and $4NR^2 = AC^2 + CD^2 + BD^2 + AB^2 - AD^2 - BC^2$ and adding these we now have

$$4(LP^2 + MQ^2 + NR^2) = AB^2 + AC^2 + AD^2 + BC^2 + BD^2 + CD^2$$

(modified from The American Mathematical Monthly, Vol. 25 Issue 3, page 122 (1918) Problem Section, Problem 522)

Note there is a vector method proof of Apollonius' Theorem in Cambridge Maths Extension 1 Year 12 Chapter 8 Review Exercise Q14 page 427 and an non-vector proof in https://en.wikipedia.org/wiki/Apollonius's_theorem

15bii. (Alternative solution 3)

Alessandro Fonda

This uses a generalisation found in 2013.

First redefine the term *bimedian* for n points as the segment joining the midpoint of one segment to the barycentre of the remaining $n-2$ points. Then the sum of the squares of the $\frac{n(n-1)}{2}$ bimedians is equal to $\frac{n}{4n-8}$ times the sum of the squares of all segments joining the n points.

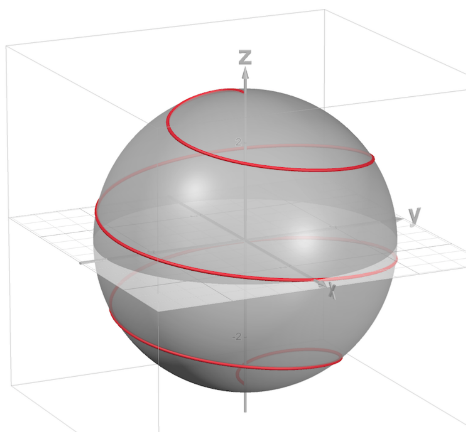
In the case $n=4$, the six bimedians coincide two by two. This explains why, in this case, we now have one half of the sum of the squares of its edges, instead of one fourth.

Hence $|\overrightarrow{AB}|^2+|\overrightarrow{AC}|^2+|\overrightarrow{AD}|^2+|\overrightarrow{BC}|^2+|\overrightarrow{BD}|^2+|\overrightarrow{CD}|^2=4(|\overrightarrow{LP}|^2+|\overrightarrow{MQ}|^2+|\overrightarrow{NR}|^2)$

Fonda, A., On a Geometrical Formula Involving Medians and Bimedians, Mathematics Magazine, Vol. 86, No. 5 (December 2013), pp. 351-357

15c. $x=\sqrt{9-t^2}\cos(\pi t), y=-\sqrt{9-t^2}\sin(\pi t), z=t$ with $-3\leq t\leq 3$ gives the equation of the curve $\mathcal{C}$

$\mathcal{C}$ is on the sphere because
$(\sqrt{9-t^2}\cos(\pi t))^2+(-\sqrt{9-t^2}\sin(\pi t))^2+t^2$
$=(9-t^2)(\cos^2(\pi t)+\sin^2(\pi t))+t^2$
$=9-t^2+t^2$
$=3^2$
Which of x, y has $+/-$ and which has sin / cos determines the shape of $\mathcal{C}$ and the one given affords the one in this desmos picture which concurs with the one in the question.



Also angular frequency of π gives the correct number of turns.

16ai. $w^3 - 1 = e^{2i\pi} - 1 = 1 - 1 = 0 = (w-1)(1+w+w^2)$ and $w \neq 1 \therefore 1 + w + w^2 = 0$

16aii. Rotate $b - c$ anticlockwise by $\frac{2\pi}{3} \Rightarrow (b-c)w = c - a$
$\therefore bw - cw - c + a = a + bw - c(1+w) = 0$
From 16ai, $1 + w = -w^2 \therefore a + bw - c(-w^2) = 0 \therefore a + bw + cw^2 = 0$

16aiii. Either $a + bw + cw^2 = 0$ or $a + bw^2 + cw = 0 \therefore (a + bw + cw^2)(a + bw^2 + cw) = 0$

From 16ai, $w^3 = 1$ and $w + w^2 = -1$

$\therefore a^2 + abw^2 + acw + bwa + b^2w^3 + bcw^2 + cw^2a + cbw^4 + c^2w^3$
$= a^2 + b^2 + c^2 + ab(w + w^2) + bc(w + w^2) + ca(w + w^2)$
$= a^2 + b^2 + c^2 + ab(-1) + bc(-1) + ca(-1)$
$= 0$
$\therefore a^2 + b^2 + c^2 = ab + bc + ca$

16bi. If $y = f(x) = x - \ln x$ then $f'(x) = 1 - \frac{1}{x} = 1 - x^{-1} = 0$ for stationary points $\therefore x = 1, y = 1$ and $f''(x) = x^{-2}$ so $f''(1) = 1 > 0$. Furthermore $f''(x) > 0 \; \forall x > 0$.

Hence $(1, 1)$ is a global minimum turning point and so for all $x > 0, f(x) \geq 1 > 0 \therefore$ $x > \ln x$ for all $x > 0$.

16bii. From 16bi, $k > \ln k \; \forall k \in \mathbb{Z}^+ \therefore \sum_{k=1}^{n} k > \sum_{k=1}^{n} \ln k \therefore \frac{n(n+1)}{2} > \ln \prod_{k=1}^{n} k = \ln n!$

Hence $n^2 + n > 2 \ln n! = \ln((n!)^2)$.

As e^x is a strictly monotonically increasing function, $e^{n^2+n} > (n!)^2$

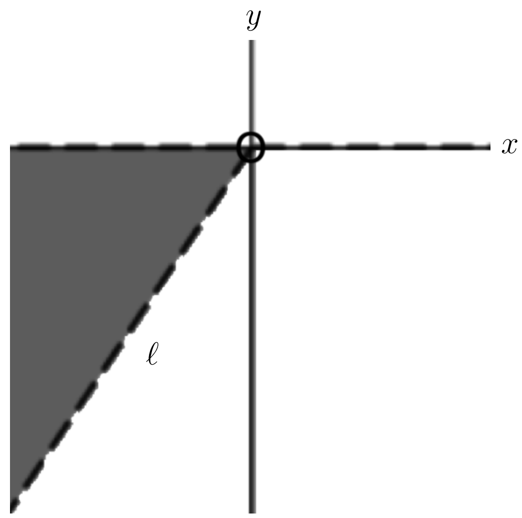
16c. $x, y \in \mathbb{R}, z, w \in \mathbb{C}, \theta = \text{Arg}(\frac{z}{w}) = -\text{Arg}(\frac{w}{z}), |z| = |w| = |\frac{z}{w}| = |\frac{w}{z}| = 1, \frac{\pi}{2} < \theta < \pi$

$\Rightarrow \frac{xz+yw}{z} = x + y \cdot \frac{w}{z} = x + y(\cos(-\theta) + i \sin(-\theta)) = x + y\cos\theta - iy\sin\theta$

and so with $\frac{\pi}{2} < \text{Arg}(\frac{xz+yw}{z}) < \pi, -y\sin\theta > 0$ and $x + y\cos\theta < 0$

So with $-1 < \cos\theta < 0 < \sin\theta < 1$ we have $y < 0$ and $x < -y\cos\theta < 0$ and hence $-x\sec\theta < y < 0$. Alternatively, since $\cos\theta = \Re(\frac{z}{w}), -\frac{x}{\Re(\frac{z}{w})} < y < 0$

If $m = -\sec\theta$ then $m > 1$ and the region satisfied by (x, y) with the condition that $\frac{\pi}{2} < \text{Arg}(\frac{xz+yw}{z}) < \pi$ will be below the line $y = 0$ and above the line $\ell : y = mx$



y
x
ℓ