HSC 2024 Mathematics Extension 2 Solutions

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1. D 2. C 3. A 4. B 5. D 6. A 7. C 8. A 9. B 10. D

1. Method 1.

$(-\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}) \cdot (3\underset{\sim}{i} + 2\underset{\sim}{j} - 5\underset{\sim}{k}) = -1 \times 3 - 1 \times 2 + 1 \times -5 = -10 \neq 0 \therefore$ answer is not A

$(\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}) \cdot (3\underset{\sim}{i} + 2\underset{\sim}{j} - 5\underset{\sim}{k}) = 1 \times 3 + 1 \times 2 - 1 \times -5 = 10 \neq 0 \therefore$ answer is not B

$(-2\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}) \cdot (3\underset{\sim}{i} + 2\underset{\sim}{j} - 5\underset{\sim}{k}) = -2 \times 3 + 3 \times 2 + 1 \times -5 = -5 \neq 0 \therefore$ answer is not C

Answer is not A, B nor C, hence the answer is D

Method 2.

$(3\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}) \cdot (3\underset{\sim}{i} + 2\underset{\sim}{j} - 5\underset{\sim}{k}) = 3 \times 3 - 2 \times 2 + 1 \times -5 = 0 \therefore$ the answer is D

2. Method 1

A is $\exists\ \theta \in (\frac{\pi}{2}, \pi) : \forall \phi \in (\pi, \frac{3\pi}{2}) \sin\theta = -\cos\phi \therefore$ answer is not A

B is $\exists\ \phi \in (\pi, \frac{3\pi}{2}) : \forall \theta \in (\frac{\pi}{2}, \pi) \sin\theta = -\cos\phi \therefore$ answer is not B

D is $\forall \phi \in (\pi, \frac{3\pi}{2})\ \exists\ \theta \in (\frac{\pi}{2}, \pi) : \sin\theta = -\cos\phi \therefore$ answer is not D

Answer is not A, B nor D and hence the answer is C.

Method 2

C is $\forall \theta \in (\frac{\pi}{2}, \pi)\ \exists\ \phi \in (\pi, \frac{3\pi}{2}) : \sin\theta = -\cos\phi \therefore$ answer is C

3. Method 1

Converse of B is 'If a polygon is not a square, then it is a rectangle' $\therefore$ answer is not B

Converse of C is 'If a polygon is not a square then it is not a rectangle' $\therefore$ answer is not C

Converse of D is 'If a polygon is not a rectangle, then it is not a square' $\therefore$ answer is not D.

Answer is not B, C nor D and hence the answer is A

Method 2.

The converse of A is 'If a polygon is a square, then it is a rectangle' $\therefore$ the answer is A

4. By the conjugate root theorem the other root is $2 - i$ and so we observe

Method 1.

Sum of roots is $3+2+i+2-i = 7$ and so the answer is not C nor D and the product of the roots is $3(2+i)(2-i) = 15$ and so the answer is not A

Answer is not A, C nor D and hence the answer is B

Method 2.

Sum of roots is $3+2+i+2-i = 7$, sum of roots 2-at-a-time is $3(2+i)+3(2-i)+(2+i)(2-i) = 6+3i+6-3i+4+1 = 17$ and the product of the roots is $3(2+i)(2-i) = 15$ and hence the answer is B

Method 3.

Some calculators have complex number functionality where we calculate that

For A, $f(2+i) = -38 - 34i \neq 0 \therefore$ the answer is not A

For C, $f(2+i) = 4 + 22i \neq 0 \therefore$ the answer is not C

For D, $f(2+i) = 42 + 56i \neq 0 \therefore$ the answer is not D

The answer is not A, C nor D and hence the answer is B

Method 4.

For B, $f(3) = f(2+i) = f(2-i) = 0$ and hence the answer is B

Method 5.

$(x-3)(x-2-i)(x-2+i) = (x-3)(x^2-4x+5) = x^3 - 7x^2 + 17x - 15 \therefore$ the answer is B

5. Method 1.

$x = -8\sin\frac{\pi t}{5}$ and $\dot{x} = -\frac{8\pi}{5}\cos\frac{\pi t}{5}$ and so for A, $\frac{\pi t}{5} = (2n+1)\pi$ for some $n \in \mathbb{Z}$ and so when $t = 7.5, n = 0.25 \notin \mathbb{Z}$ $\therefore$ answer is not A

for B, $\frac{\pi t}{5} = 2\pi n$ for some $n \in \mathbb{Z}$ and so when $t = 7.5, n = 0.75 \notin \mathbb{Z}$ $\therefore$ answer is not B

for C, $\frac{\pi t}{5} = \frac{(4n+1)\pi}{2}$ for some $n \in \mathbb{Z}$ and so when $t = 7.5, n = 0.5 \notin \mathbb{Z}$ $\therefore$ answer is not C

The answer is not A, B, not C and hence the answer is D

Method 2.

For D, $-8\sin\frac{7.5\pi}{5} = 8$ and $-\frac{8\pi}{5}\cos\frac{7.5\pi}{5} = 0$ and hence the answer is D

6. Method 1.

Replace the 9 km mass with m kg mass. Now

for B, $mg - T = mg, T - 5g = 5g \therefore m - 5 = m + 5$ no solution $\therefore m \neq 9 \therefore$ answer is not B

for C, $mg - T = \frac{7mg}{2}, T - 5g = \frac{35g}{2} \therefore m - 5 = \frac{7m+35}{2} \therefore m = -9 \neq 9 \therefore$ answer is not C

for D, $mg - T = 4mg, T - 5g = 20g \therefore m - 5 = 4m + 20 \therefore m = -\frac{25}{3} \neq 9 \therefore$ answer is not D

Method 2.

$9g - T = 9a, T - 5g = 5a \therefore 4g = 14a \therefore a = \frac{2g}{7} \therefore$ the answer is A

7. Method 1

Consider the complex number $z = (1+\sqrt{2})(1-i)$ on the circle

Then $|z| = 2+\sqrt{2}$ and so $|z| > \sqrt{2}, |z| > \sqrt{10}$ and $|z| > 2-\sqrt{2}$ and so the answer is not A, B nor D and hence the answer is C.

Method 2.

$|z| = |z - 1 + i + 1 - i| \leq |z - 1 + i| + |1 - i| = 2 + \sqrt{2}$ by the triangle inequality with equality iff $z = (1+\sqrt{2})(1-i)$ hence the answer is C

8. Method 1.

$|e^{-z}| = e^{-x} \neq e^{x} = |e^{\bar{z}}| \therefore e^{-z} \neq e^{\bar{z}} \therefore$ not B

$|e^{2x}e^{z}| = e^{3x} \neq e^{x} = |e^{\bar{z}}| \therefore e^{2x}e^{z} \neq e^{\bar{z}} \therefore$ not C

$|e^{-2x}e^{z}| = e^{-x} \neq e^{x} = |e^{\bar{z}}| \therefore e^{-2x}e^{z} \neq e^{\bar{z}} \therefore$ not D

not B,C nor D $\Rightarrow$ answer is A.

Method 2

$|\overline{e^{z}}| = |e^{z}| = e^{x} = |e^{\bar{z}}|$ and $\arg \overline{e^{z}} = -y + 2\pi k = \arg e^{\bar{z}} \therefore \overline{e^{z}} = e^{\bar{z}} \therefore$ the answer is A

9. Method 1.

Solutions are $\sqrt{3}e^{\frac{(2k+1)\pi i}{4}}$ for $k = -2, -1, 0, 1$ and for positive principal argument $k = 0, 1$ and hence their product is -3 and so answer is not A, C nor D and hence the answer is B

Method 2

The product is -3 and hence the answer is B

10. Method 1.

$$
\begin{aligned}
\cos\theta &= \frac{\underset{\sim}{a}\cdot(\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c})}{|\underset{\sim}{a}||\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}|} \\
&= \frac{\underset{\sim}{a}\cdot\underset{\sim}{a}+\underset{\sim}{a}\cdot\underset{\sim}{b}+\underset{\sim}{a}\cdot\underset{\sim}{c}}{1\sqrt{|\underset{\sim}{a}|^2+|\underset{\sim}{b}|^2+|\underset{\sim}{c}|^2+2(\underset{\sim}{a}\cdot\underset{\sim}{b}+\underset{\sim}{a}\cdot\underset{\sim}{c}+\underset{\sim}{b}\cdot\underset{\sim}{c})}} \\
&= \frac{1+0+\underset{\sim}{a}\cdot\underset{\sim}{c}}{\sqrt{1+1+1+2(0+\underset{\sim}{a}\cdot\underset{\sim}{c}+0)}} \\
&= \frac{1+|\underset{\sim}{a}||\underset{\sim}{c}|\cos\alpha}{\sqrt{3+2|\underset{\sim}{a}||\underset{\sim}{c}|\cos\alpha}} \text{ where } \alpha \text{ is the angle between } \underset{\sim}{a} \text{ and } \underset{\sim}{c} \\
&= \frac{1+\cos\alpha}{\sqrt{3+2\cos\alpha}} \\
&\leq \frac{2\sqrt{(1+\cos^2\alpha)/2}}{\sqrt{3+2\cos\alpha}}
\end{aligned}
$$

by the A.M.-Q.M. inequality with minimum θ for maximum $\cos\theta$ when

$1 + \cos\alpha = 2\sqrt{(1+\cos^2\alpha)/2} \therefore 1 + 2\cos\alpha + \cos^2\alpha = 2 + 2\cos^2\alpha$

$\therefore \cos^2\alpha - 2\cos\alpha + 1 = (\cos\alpha - 1)^2 = 0 \therefore \cos\alpha = 1$

For A, $\cos\theta = 0 \therefore \cos\alpha = -1 \neq 1 \therefore$ the answer is not A

For B, $\cos\theta = \frac{1}{\sqrt{3}} \therefore 3(1+\cos\alpha)^2 = 3 + 2\cos\alpha \therefore \cos\alpha(3\cos\alpha + 4) = 0$ and $\cos\alpha = 0 \neq 1 \therefore$ the answer is not B

For C, $\cos\theta = \frac{1}{\sqrt{2}} \therefore 2(1+\cos\alpha)^2 = 3 + 2\cos\alpha \therefore 2\cos^2\alpha + 2\cos\alpha - 1 = 0$ and $\cos\alpha = \frac{\sqrt{3}-1}{2} \neq 1 \therefore$ the answer is not C.

The answer is not A, B, nor C hence the answer is D

Method 2.

$\cos\alpha = 1$ to minimise θ hence $\cos\theta = \frac{1+1}{\sqrt{3+2}} = \frac{2}{\sqrt{5}}$ and hence the answer is D

11a. $\int xe^x\,dx = \int x\frac{de^x}{dx}\,dx = xe^x - \int e^x\frac{dx}{dx}\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C$

11bi. $z + \bar{w} = 2 + 3i + 1 + 5i = 3 + 8i$

11bii. $z^2 = 4 + 12i - 9 = -5 + 12i$

11c. $\cos^{-1}\frac{1\times4+2\times-4-2\times7}{\sqrt{(1+4+4)(16+16+49)}} \approx 2.3$

11d. Where $t = \tan\frac{\theta}{2}$, $dt = \frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta = \frac{1}{2}(1+t^2)\,d\theta$ so $d\theta = \frac{2\,dt}{1+t^2}$ and $\sin\theta = \frac{2t}{1+t^2}$

Also, when $\theta = 0, t = 0$ and when $\theta = \frac{\pi}{2}, t = 1$

$$\begin{aligned}\therefore \int_0^{\frac{\pi}{2}} \frac{1}{\sin\theta+1}\,d\theta &= \int_0^1 \frac{1}{\frac{2t}{1+t^2}+1}\cdot\frac{2}{1+t^2}\,dt\\ &= \int_0^1 \frac{2}{(1+t)^2}\,dt\\ &= [\tfrac{-2}{1+t}]_0^1\\ &= -1+2\\ &= 1\end{aligned}$$

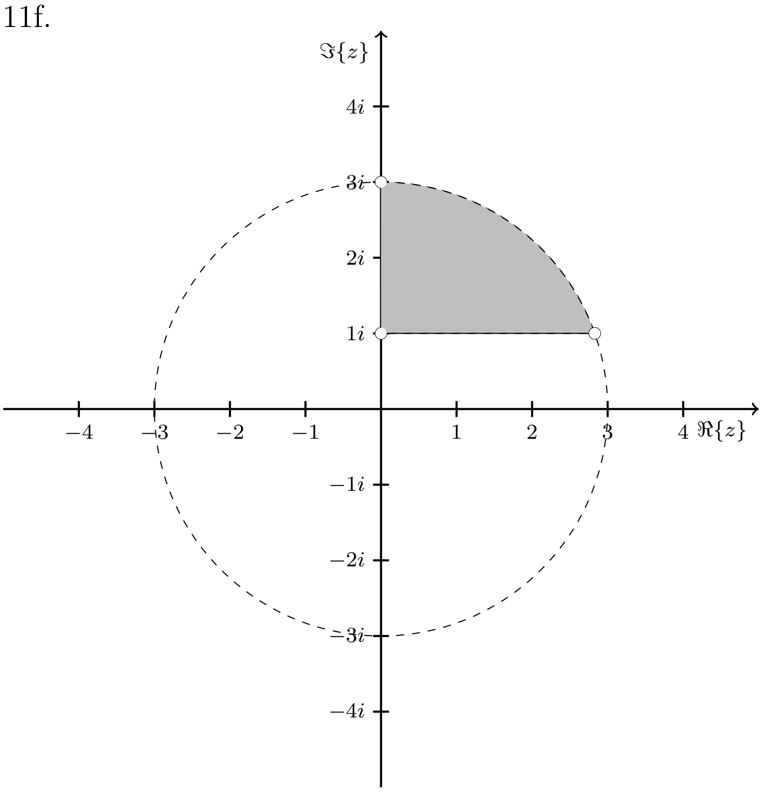
11ei. $\sqrt{3} + i = \sqrt{\sqrt{3}^2 + 1^2}(\cos\tan^{-1}\frac{1}{\sqrt{3}} + i\sin\tan^{-1}\frac{1}{\sqrt{3}}) = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$

11ei. Alternative solution, by calculator, $\sqrt{3} + i = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$

11eii. $(\sqrt{3}+i)^7 = 2^7(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})^7 = 128(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}) = 128(-\frac{\sqrt{3}}{2} - \frac{1}{2}) = -64\sqrt{3} - 64i$

11eii. Alternative solution by calculator, $(\sqrt{3}+i)^7 = -64\sqrt{3} - 64i$

11f.



12ai. $\frac{1\times2+2\times0+3\times-4}{2^2+0^2+4^2}\begin{pmatrix}2\\0\\-4\end{pmatrix}=\begin{pmatrix}-1\\0\\2\end{pmatrix}$

12aii. $\left(\begin{pmatrix}1\\2\\3\end{pmatrix}-\begin{pmatrix}-1\\0\\2\end{pmatrix}\right)\cdot\begin{pmatrix}2\\0\\-4\end{pmatrix}=\begin{pmatrix}2\\2\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\0\\2\end{pmatrix}=2\times-1+2\times0+1\times2=0$
$\therefore \underset{\sim}{a}-\frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\underset{\sim}{b}\cdot\underset{\sim}{b}}\underset{\sim}{b}\perp\underset{\sim}{b}$

12b. $\frac{3x^2+2x+1}{(x-1)(x^2+1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+1}$
$\Rightarrow 3x^2+2x+1=A(x^2+1)+(Bx+C)(x-1)=(A+B)x^2+(C-B)x+A-C$

$A+B=3, C-B=2$ and $A-C=1$ so $A=3-B\therefore 3-B-C=1$

Now $C-B+3-B-C=3-2B=2+1$ hence $B=0, A=3, C=2$

Alternatively,

$\Rightarrow\begin{pmatrix}1&1&0\\0&-1&1\\1&0&-1\end{pmatrix}\begin{pmatrix}A\\B\\C\end{pmatrix}=\begin{pmatrix}3\\2\\1\end{pmatrix}\Rightarrow\begin{pmatrix}A\\B\\C\end{pmatrix}=\begin{pmatrix}1&1&0\\0&-1&1\\1&0&-1\end{pmatrix}^{-1}\begin{pmatrix}3\\2\\1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1&1&1\\1&-1&-1\\1&1&-1\end{pmatrix}\begin{pmatrix}3\\2\\1\end{pmatrix}=\begin{pmatrix}3\\0\\2\end{pmatrix}$

$\Rightarrow\int\frac{3x^2+2x+1}{(x-1)(x^2+1)}\,dx=\int(\frac{3}{x-1}+\frac{2}{x^2+1})\,dx=3\ln|x-1|+2\tan^{-1}x+C$

12ci. $|z|$ is real, so $a+8+(b+12)i$ is real so $b+12=0$ so $b=-12$

12cii. $a^2+12^2=(a+8)^2\therefore a^2+144=a^2+16a+64\therefore 16a=80\therefore a=5$ and $z=5-12i$

12cii. Alternative solution

The question is equivalent to this. Given $A=(0,0), B=(a,b), C=(\sqrt{a^2+b^2},0)$, $D=(8,12)$, find a and b such that $ABCD$ is a parallelogram. Hereby, as diagonals AC and BD bisect each other, now $(\frac{\sqrt{a^2+b^2}}{2},0)=(\frac{a+8}{2},\frac{b+12}{2})$ so $b=-12$ and $a^2+144=a^2+16a+64$ and $a=5$. Hence $z=5-12i$.

12d. n odd or even $\Rightarrow (n+1)^{41}-79n^{40}$ is odd and so cannot be 2

12ei. $\underset{\sim}{v}=3\underset{\sim}{i}+5\underset{\sim}{j}-4\underset{\sim}{k}+\lambda((7-3)\underset{\sim}{i}+(0-5)\underset{\sim}{j}+(2+4)\underset{\sim}{k})=3\underset{\sim}{i}+5\underset{\sim}{j}-4\underset{\sim}{k}+\lambda(4\underset{\sim}{i}-5\underset{\sim}{j}+6\underset{\sim}{k})$

12eii. If C lies on ℓ then $3+4\lambda=10 \therefore \lambda=\frac{7}{4}$ but $5-5(\frac{7}{4})=-\frac{15}{4}\neq 5 \therefore C$ does not lie on ℓ

13ai.$|AB|^2=(p-8)^2+(p+6)^2+(2p-5)^2=p^2-16p+64+p^2+12p+36+4p^2-20p+25=6p^2-24p+125$

13aii. $\frac{d|AB|^2}{dp}=12p-24=0\Rightarrow p=2\Rightarrow |AB|^2=6(2)^2-24(2)+125=101$ and $\frac{d^2|AB|^2}{dp^2}=12>0 \therefore$ minimum distance $=\sqrt{101}$

13aii. Alternative solution using cross product

Last year there was a question in the Extension 1 paper which could be solved more efficiently using cross product. Because the question said "or otherwise" the markers said the cross product method would be accepted.

Likewise in this question it said "or otherwise" and we proceed thusly

$$\frac{|(8\underset{\sim}{i}-6\underset{\sim}{j}+5\underset{\sim}{k})\times(\underset{\sim}{i}+\underset{\sim}{j}+2\underset{\sim}{k})|}{|\underset{\sim}{i}+\underset{\sim}{j}+2\underset{\sim}{k}|}=\frac{\left\|\begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ 8 & -6 & 5\\ 1 & 1 & 2\end{vmatrix}\right\|}{\sqrt{1^2+1^2+2^2}}=\frac{|-17\underset{\sim}{i}-11\underset{\sim}{j}+14\underset{\sim}{k}|}{\sqrt{6}}=\frac{\sqrt{17^2+11^2+14^2}}{\sqrt{6}}=\sqrt{101}$$

13b. $\ddot{x}=-n^2(x-c)\Rightarrow \dot{x}^2=n^2(a^2-(x-c)^2)$ and if $x=0,|\dot{x}|=v$ then $v^2=n^2(a^2-c^2)\Rightarrow a=\frac{\sqrt{n^2c^2+v^2}}{n}$ and so with $v=4,n=2,c=-1$, distance travelled in first period is $4a=\frac{4\sqrt{2^2(-1)^2+4^2}}{2}=4\sqrt{5}$

13ci. $\frac{d}{dx}\frac{1}{2}v^2=-kv^2 \therefore \int_{V=40}^{V=v}\frac{d(V^2)}{V^2}=[\ln V^2]_{40}^{v}=2\ln\frac{v}{40}=\int_{X=0}^{X=x}-2kdX=-2k[X]_0^x=-2kx \therefore v=40e^{-kx}$

13cii. $10=40e^{-15k} \therefore e^{15k}=4 \therefore 15k=\ln 4 \therefore k=\frac{\ln 4}{15}$

13ciii. $\frac{dv}{dt}=-kv^2$

$\therefore \int_{40}^{30} -v^{-2}\,dv = [v^{-1}]_{40}^{30}$
$= \frac{1}{30} - \frac{1}{40}$
$= \frac{1}{120}$
$= \int_0^T k\,dT$
$= [kT]_0^t$
$= kt$
$= \frac{t\ln 4}{15} \therefore t = \frac{15}{120\ln 4} = \frac{1}{8\ln 4}$ seconds

13d. $a\sqrt{bc}+b\sqrt{ac}+c\sqrt{ab} \leq \frac{a}{2}(b+c)+\frac{b}{2}(a+c)+\frac{c}{2}(a+b) = ab+bc+ca = (\frac{1}{a}+\frac{1}{b}+\frac{1}{c})abc = abc$

14a. $a = 2k+1$ for an integer $k \Rightarrow a^2-1 = (2k+1)^2-1 = 4k^2+4k = 4k(k+1)$ and k or $k+1$ is even $\Rightarrow 8|a^2-1$

14bi. ${}^{10}C_5 = 252 < 256 = 2^8 \therefore$ it is true for $n = 5$

If it is true for $n = k$ then ${}^{2k}C_k < 2^{2k-2}$ hence

$${}^{2k+2}C_{k+1} = \frac{(2k+2)!}{((k+1)!)^2}$$
$$= \frac{(2k+2)(2k+1)(2k)!}{(k+1)^2(k!)^2}$$
$$= \frac{2(2k+2-1)}{k+1} \cdot {}^{2k}C_k$$
$$= 2(2-\frac{1}{k+1}) \cdot {}^{2k}C_k$$
$$\leq 2^2 \cdot 2^{2k-2}$$
$= 2^{2k}$ and so it is true for $n = k+1$

Hence by the principle of mathematical induction it is true for all positive integers $n \geq 5$

14c. $\arg(w) - \arg(z) = \frac{\pi}{2}$ so $w = riz$ for some real number r and so
$\left|\frac{z-w}{z+w}\right| = \left|\frac{z-riz}{z+riz}\right| = \frac{\sqrt{1+r^2}}{\sqrt{1+r^2}} = 1$

14d. $\int \frac{1}{x}\,dx = \ln|x|+c_1$ and $1+\int \frac{1}{x}\,dx = 1+\ln|x|+c_2$ for constants c_1, c_2 so subtracting the integral from both sides, $c_1 - c_2 = 1$ not $0 = 1$.

14ei. $\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR} = \frac{3}{5}\underset{\sim}{a} + k(3\underset{\sim}{b} - \frac{3}{5}\underset{\sim}{a}) = \frac{3}{5}(1-k)\underset{\sim}{a} + 3k\underset{\sim}{b}$

14eii. $(1-h)\underset{\sim}{a}+h\underset{\sim}{b} = \frac{3}{5}(1-k)\underset{\sim}{a}+3k\underset{\sim}{b} \therefore 1-h = \frac{3}{5}(1-k)$ and $h = 3k \therefore 1-3k = \frac{3}{5}(1-k)$ $\therefore 5-15k = 3-3k \therefore 12k = 2 \therefore k = \frac{1}{6}$

14ciii. $\overrightarrow{OR} = \frac{1}{2}(\underset{\sim}{a}+\underset{\sim}{b})$ so $\overrightarrow{OT} = \mu(\underset{\sim}{a}+\underset{\sim}{b})$ for some real $\mu > 0$

Also, $\overrightarrow{OT} = \underset{\sim}{a} + \lambda(3\underset{\sim}{b} - \underset{\sim}{a})$ for some real $\lambda > 0 \therefore 1-\lambda = \mu$ and $3\lambda = \mu \therefore 1 - \frac{\mu}{3} = \mu$

$\therefore \mu = \frac{3}{4}$. Hence $\overrightarrow{OT} = \frac{3}{4}(\underset{\sim}{a} + \underset{\sim}{b})$

15ai. $\overrightarrow{AM} = \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b}) - \underset{\sim}{a} = \frac{1}{2}(\underset{\sim}{b} - \underset{\sim}{a}) = \frac{1}{2}\overrightarrow{AB} \therefore A, M, B$ are collinear $\therefore M$ is on the line joining AB

15aii. $\overrightarrow{MC} = \underset{\sim}{c} - \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b}) = \frac{1}{2}(2\underset{\sim}{c} - \underset{\sim}{a} - \underset{\sim}{b})$ and
$\overrightarrow{MG} = \frac{1}{3}(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}) - \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b}) = \frac{1}{6}(2\underset{\sim}{c} - \underset{\sim}{a} - \underset{\sim}{b}) = \frac{1}{3}\overrightarrow{MC} \therefore M, G, C$ are collinear and G divides MC in ratio $1:2$ so G lies on the line passing through M and C and lies between M and C.

15aiii. x, w, z are on unit circle and are unequal so $m = \frac{1}{2}(x + w)$ lies strictly between x, w and $g = \frac{1}{3}(x + w + z)$ lies strictly between m and z, hence g is strictly inside the unit circle, hence $|g| < 1$. $|xwz| = |x| = |w| = |z| = 1$ and cube roots of xwz have modulus 1 and so g cannot be a cube root of xwz.

15aiii. Alternative method

By triangle inequality, $|\frac{1}{3}(x + w + z)| \leq \frac{1}{3}(|x| + |w| + |z|) = \frac{1}{3}(1 + 1 + 1) = 1$ with equality if and only if x, w and z are non-negative scalar multiples of x, w or z but they are not since they are unequal and on the unit circle, hence $|\frac{1}{3}(x + w + z)| < 1$ and since $|xwz| = 1$ cube roots of xwz have modulus 1 then $\frac{1}{3}(x + w + z)$ cannot be a cube root of xwz.

15b.
$$\begin{aligned}
I_n &= \int_0^a x^{n+\frac{1}{2}}(a - x)^{\frac{1}{2}}\,dx \\
&= \int_0^a x^{n+\frac{1}{2}} \frac{d}{dx}(-\tfrac{2}{3}(a - x)^{\frac{3}{2}})\,dx \\
&= [-\tfrac{2}{3}x^{n+\frac{1}{2}}(a - x)^{\frac{3}{2}}]_0^a - \int_0^a -\tfrac{2}{3}(a - x)^{\frac{3}{2}} \frac{d}{dx}(x^{n+\frac{1}{2}})\,dx \\
&= 0 - 0 + \tfrac{2}{3}(n + \tfrac{1}{2}) \int_0^a x^{n-\frac{1}{2}}(a - x)(a - x)^{\frac{1}{2}}\,dx \\
&= \tfrac{2n+1}{3}(a \int_0^a x^{n-\frac{1}{2}}(a - x)^{\frac{1}{2}}\,dx - \int_0^a x^{n+\frac{1}{2}}(a - x)^{\frac{1}{2}})\,dx) \\
&= \tfrac{(2n+1)aI_{n-1}}{3} - \tfrac{(2n+1)I_n}{3}
\end{aligned}$$
$\Rightarrow \frac{(2n+4)I_n}{3} = \frac{(2n+1)aI_{n-1}}{3} \therefore$ for $n > 0, (2n + 4)I_n = a(2n + 1)I_{n-1}$

15b. Alternative solution. Where B is the beta function

$I_n = a^{n+2}B(n+\frac{3}{2}, \frac{3}{2}) = a^{n+2} \cdot \frac{n+\frac{1}{2}}{n+\frac{1}{2}+\frac{3}{2}} B(n+\frac{1}{2}, \frac{3}{2}) = a \cdot \frac{2n+1}{2n+4} a^{n+1} B(n+\frac{1}{2}, \frac{3}{2}) = a \cdot \frac{2n+1}{2n+4} I_{n-1}$
$\Rightarrow (2n + 4)I_n = a(2n + 1)I_{n-1}$

15ci.
$$\begin{aligned}
v^2 &= 2\int_0^v V\,dV \\
&= 2\int_6^x (27gX^{-3} - g)\,dX \\
&= [-27gX^{-2} - 2gX]_6^x \\
&= -27gx^{-2} - 2gx - (-\tfrac{27g}{36} - 12g) \\
&= g(\tfrac{51}{4} - 2x - \tfrac{27}{x^2})
\end{aligned}$$

15cii. $v^2 = \frac{g(x-6)(8x^2-3x-18)}{4x^2} = 0$ and $0 < x < 6 \Rightarrow x = \frac{3+\sqrt{3^2+4\times 8\times 18}}{16} = \frac{3+\sqrt{585}}{16} \approx 1.7$

15d. For $0 < x < 2$, let $\theta = \sin^{-1}(x-1) \therefore x = \sin\theta + 1$ and $dx = \cos\theta$ and so

$$\begin{aligned}\int \frac{2x^2}{\sqrt{2x-x^2}}\,dx &= \int \frac{2(\sin\theta+1)^2\cos\theta}{\sqrt{1-\sin^2\theta}}\,d\theta \\ &= \int(2\sin^2\theta + 4\sin\theta + 2)\,d\theta \\ &= \int(3 - \cos 2\theta + 4\sin\theta)\,d\theta \\ &= 3\theta - \tfrac{1}{2}\sin 2\theta - 4\cos\theta + C \\ &= 3\theta - (\sin\theta + 4)\cos\theta + C \\ &= 3\sin^{-1}(x-1) - (x+3)\sqrt{2x-x^2} + C\end{aligned}$$

16a. $\frac{b}{a}(\frac{d}{dx}\cos kx|_{x=a}) = \frac{-bk\sin ak}{a} = -1$ and also for $0 < x < \frac{\pi}{k}, 0 < \cos kx < 1$ and if $f(x) = x - \sin x, f'(x) = 1 - \cos x$ and so $0 < f'(x) < 1$ for $0 < x < \pi$ and so $f(x) > f(0) \therefore x > \sin x$

Now $\sin ak = \frac{a}{bk} < ak$ so $k^2 > \frac{1}{b} > 1$ and as $k > 0, k > 1$

16bi.
$$\begin{aligned}(\gamma+\bar{\gamma})^3 - 3(\gamma+\bar{\gamma}) + 1 &= \gamma^3 + 3\gamma^2\bar{\gamma} + 3\gamma\bar{\gamma}^2 + \bar{\gamma}^3 - 3\gamma - 3\bar{\gamma} + 1 \\ &= w + \bar{w} + 3\gamma\bar{\gamma}(\gamma+\bar{\gamma}) - 3\gamma - 3\bar{\gamma} + 1 \\ &= 2\cos\tfrac{2\pi}{3} + 3|\gamma|^2(\gamma+\bar{\gamma}) - 3\gamma - 3\bar{\gamma} + 1 \\ &= -1 + 3\gamma + 3\bar{\gamma} - 3\gamma - 3\gamma + 1 \text{ as } |\gamma| = 1 \\ &= 0\end{aligned}$$

Hence $\gamma + \bar{\gamma}$ is a real root of $z^3 - 3z + 1 = 0$

16bii. $\gamma = w^{\frac{1}{3}} = (e^{\frac{2\pi i}{3}+2\pi ki})^{\frac{1}{3}} = e^{\frac{2\pi i(3k+1)}{9}}$ for $k = 0, \pm 1$ and so $\gamma + \bar{\gamma} = 2\cos\frac{2\pi(3k+1)}{9}$ are roots of $z^3 - 3z + 1 = 0$ and so product of roots is
$2\cos\frac{2\pi}{9}\cdot 2\cos\frac{8\pi}{9}\cdot 2\cos\frac{-4\pi}{9} = 8\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\cos\frac{8\pi}{9} = -1 \therefore \cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\cos\frac{8\pi}{9} = -\frac{1}{8}$.

So $\cos\frac{2^n\pi}{9}\cos\frac{2^{n+1}\pi}{9}\cos\frac{2^{n+2}\pi}{9} = -\frac{1}{8}$ is true for $n = 1$

If it is true for $n = k$ then $\cos\frac{2^k\pi}{9}\cos\frac{2^{k+1}\pi}{9}\cos\frac{2^{k+2}\pi}{9} = -\frac{1}{8}$ and since

$\cos\frac{2^{k+3}\pi}{9} - \cos\frac{2^k\pi}{9} = -2\sin\frac{1}{2}(\frac{2^{k+3}\pi}{9} + \frac{2^k\pi}{9})\sin\frac{1}{2}(\frac{2^{k+3}\pi}{9} - \frac{2^k\pi}{9}) = -2\sin 2^{k-1}\pi\sin\frac{7(2^{k-1})\pi}{9} = 0$ for $k \geq 1$ then $\cos\frac{2^{k+1}\pi}{9}\cos\frac{2^{k+2}\pi}{9}\cos\frac{2^{k+3}\pi}{9} = \cos\frac{2^k\pi}{9}\cos\frac{2^{k+1}\pi}{9}\cos\frac{2^{k+2}\pi}{9} = -\frac{1}{8}$ and so it is true for $n = k+1$

Hence by the principal of mathematical induction, $\cos\frac{2^n\pi}{9}\cos\frac{2^{n+1}\pi}{9}\cos\frac{2^{n+2}\pi}{9} = -\frac{1}{8}$ for $n \geq 1$

16c. If x_A = displacement of A and x_B = displacement of B and $x = x_A - x_B$ then $\ddot{x} = -g - k\dot{x}_A + g + k\dot{x}_B = -k(\dot{x}_A - \dot{x}_B) = -k\dot{x} \therefore \dot{x} = -2v_0e^{-kt}$ so

$$
\begin{aligned}
\int_d^0 -1\,dx &= [-x]_d^0 \\
&= 0 + d \\
&= d \\
&= \int_0^t 2v_0 e^{-kT}\,dT \\
&= [-\tfrac{2v_0}{k} e^{-kT}]_0^t \\
&= \tfrac{2v_0}{k}(1 - e^{-kt}) \\
1 - e^{-kt} &= \tfrac{kd}{2v_0} \\
e^{-kt} &= 1 - \tfrac{kd}{2v_0} \\
&= \tfrac{2v_0 - kd}{2v_0} \\
e^{kt} &= \tfrac{2v_0}{2v_0 - kd} \\
kt &= \ln \tfrac{2v_0}{2v_0 - kd} \\
t &= \tfrac{1}{k} \ln \tfrac{2v_0}{2v_0 - kd}
\end{aligned}
$$