Student Number

Hunters Hill
High School

2024 **TRIAL EXAMINATION**

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/ or calculations

Total marks: 100

Section I – 10 marks (pages 3–7)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 8–15)

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

Section I

10 Marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10

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1. Given $z = 3 - 2i$ and $w = 1 + i$, then $\dfrac{z}{w}$ in simplest form is

(A) 1

(B) $5 + i$

(C) $\dfrac{1}{\sqrt{2}} - \dfrac{5}{\sqrt{2}}i$

(D) $\dfrac{1}{2} - \dfrac{5}{2}i$

2. Consider the statement:
'If n is even, then if n is a multiple of 5, then n is a multiple of 10.'
Which of the following is the negation of this statement?

(A) n is odd and n is not a multiple of 5 or 10

(B) n is even and n is a multiple of 5 but not a multiple of 10

(C) If n is even, then n is not a multiple of 5 and n is not a multiple of 10

(D) If n is odd, then if n is not a multiple of 5 then n is not a multiple of 10

3. If $\omega \neq 1$ is a cube of unity, what is the value of $(1+\omega)^2$?

(A) ω^3

(B) ω^2

(C) ω

(D) 1

4. Which of the following is obtained when $\dfrac{1-i}{1+\sqrt{3}i}$ is expressed in modulus-argument form?

(A) $\dfrac{1}{\sqrt{2}}\operatorname{cis}\left(\dfrac{17\pi}{12}\right)$

(B) $\sqrt{2}\operatorname{cis}\left(\dfrac{17\pi}{12}\right)$

(C) $\dfrac{1}{\sqrt{2}}\operatorname{cis}\left(\dfrac{7\pi}{12}\right)$

(D) $\sqrt{2}\operatorname{cis}\left(\dfrac{7\pi}{12}\right)$

5. Which of the following is the unit vector perpendicular to $2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$?

(A) $-2\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$

(B) $\dfrac{1}{\sqrt{3}}\left(-\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}\right)$

(C) $2\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k}$

(D) $\dfrac{1}{14}\left(2\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k}\right)$

6. Which integral is necessarily equal to $\int_{-a}^{a} f(x)dx$?

(A) $\int_0^a \big(f(x) - f(-x)\big)dx$

(B) $\int_0^a \big(f(x) - f(a-x)\big)dx$

(C) $\int_0^a \big(f(x-a) + f(-x)\big)dx$

(D) $\int_0^a \big(f(x-a) + f(a-x)\big)dx$

7. The displacement x of a particle at time t is given by

$$x = 6\sin 5t + 8\cos 5t.$$

What is the maximum velocity of the particle?

(A) 5

(B) 10

(C) $10\sqrt{5}$

(D) 50

8. The points A, B and C are collinear, where

$$\overrightarrow{OA} = a\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}, \qquad \overrightarrow{OB} = -2\underset{\sim}{i} + b\underset{\sim}{j} + \underset{\sim}{k}, \qquad \overrightarrow{OC} = 3\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k}$$

What are the values of a and b?

(A) $a = 5, \quad b = -3$

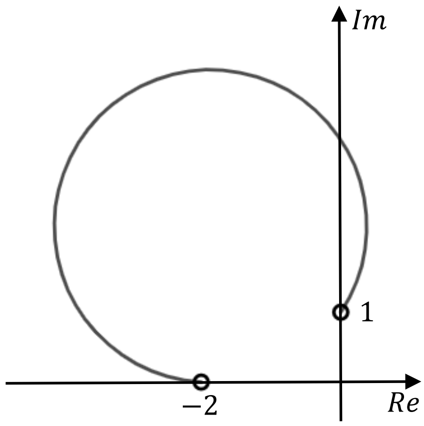
(B) $a = -4, \quad b = \dfrac{8}{7}$

(C) $a = \dfrac{-16}{3}, \quad b = \dfrac{4}{5}$

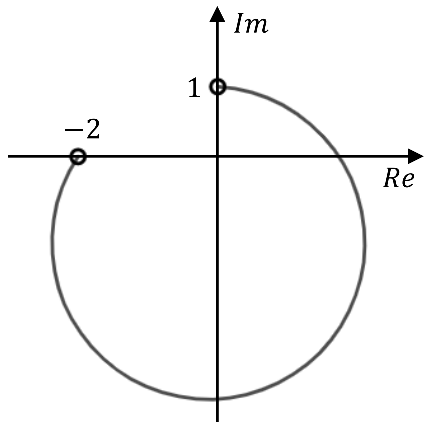
(D) $a = \dfrac{1}{3}, \quad b = 8$

9. Which of the following curves represents $\arg\left(\dfrac{z-i}{z+2}\right) = \dfrac{\pi}{3}$?

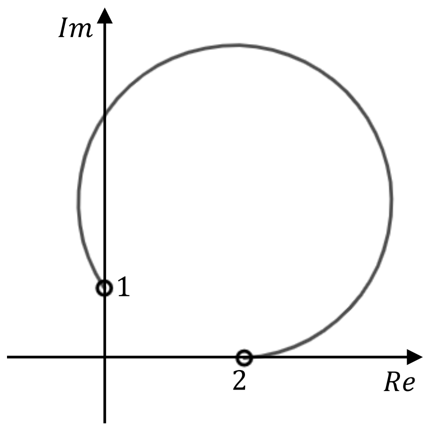
(A)



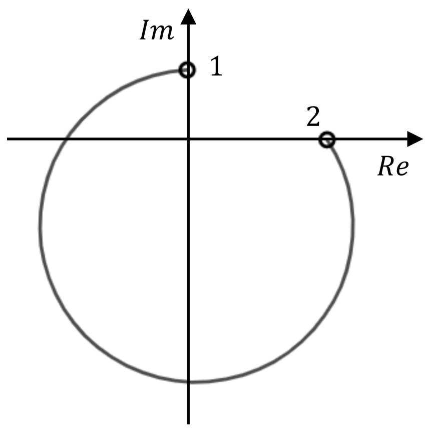
(B)



(C)



(D)



10. The acceleration, a ms^{-2}, of a particle moving in a straight line is given by $a = 8 - v^2$, where v is the velocity of the particle at any time t. The initial velocity of the particle when at the origin O is 2 ms^{-1}.

The displacement of the particle from O when its velocity is $\sqrt{3}$ ms^{-1} is

(A) $\ln\frac{1}{2}$

(B) $\frac{1}{2}\ln\left(\frac{4}{5}\right)$

(C) $\frac{1}{2}\ln\left(\frac{1}{5}\right)$

(D) $\frac{1}{2}\ln\left(\frac{5}{8}\right)$

End of Section I

Section II

90 Marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section

Begin each question in a new writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (16 marks) Begin a new Writing Booklet.

(a) Find the square roots of $15 + 8i$. **2**

(b)

i. Write $1 - \sqrt{3}\,i$ in mod-arg form. **2**

ii. Hence find $\left(1 - \sqrt{3}\,i\right)^5$ in the form $a + ib$. **3**

(c) Integrate

i. $\int \sin^3 x \, dx$ **2**

ii. $\int \frac{x^3 + 2x + 4}{x + 1} dx$ **2**

(d) Express $\frac{7x + 11}{(x - 3)(x + 1)^2}$ as a sum of partial fractions over $\mathbb{R}$. **3**

(e) The complex numbers $z = 3e^{i\frac{\pi}{3}}$ and $w = 9e^{i\frac{\pi}{4}}$ are given. **2**
Find the value of $\frac{z}{w}$, giving the answer in the form $re^{i\theta}$.

End of Question 11

Question 12 (14 marks) Begin a new Writing Booklet.

(a) If n is a positive integer, prove $\sqrt{2+6n}$ is always irrational. **3**

(b) A polynomial $P(z)$ has the equation $P(z) = z^3 - 6z^2 + kz - 26$, where $k \in \mathbb{R}$. It is known that $2 - 3i$ is a zero of $P(z)$.

i. Find the three complex roots of the polynomial, $P(z)$. **2**

ii. Hence, find the value of k. **1**

(c) Use an appropriate substitution to evaluate $\displaystyle\int_{\sqrt{7}}^{\sqrt{10}} x^3\sqrt{x^2-6}\,dx$. **3**

(d) **i.** Two people are arguing about the truth of the following statement. **1**

$$\exists x \in \mathbb{R} \text{ such that } x^2 = \sqrt{x}$$

Person A claims to prove the above statement by saying that $x = 1$.
Explain why this is sufficient proof.

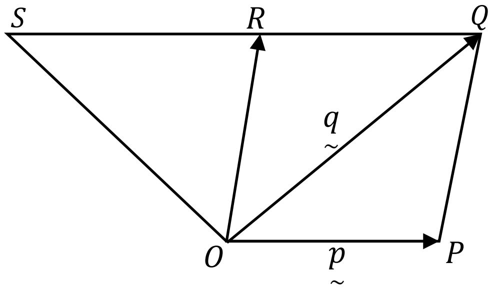
ii. Person B realises his mistake; he'd read the statement as **1**

$$\forall x \in \mathbb{R}, \qquad x^2 = \sqrt{x}$$

Explain how Person B might have disproved this new statement.

(e) $OPQS$ is a trapezium in which $\overrightarrow{OP} = \underset{\sim}{p}$ and $\overrightarrow{OQ} = \underset{\sim}{q}$. **3**

OR is parallel to PQ and $SR : RQ = 2 : 3$.



Using vectors, express $\overrightarrow{SP}$ in terms of $\underset{\sim}{p}$ and $\underset{\sim}{q}$.

End of Question 12

Question 13 (15 marks) Begin a new Writing Booklet.

(a) Find the 5th roots of $32i$. **3**

(b) Integrate $\displaystyle\int \sin^3\theta\cos^4\theta\, d\theta$. **3**

(c) A sequence is defined by **3**

$$a_1 = 5, a_2 = 13 \text{ and } a_n = 5a_{n-1} - 6a_{n-2} \text{ for integers } n \geq 3.$$

Use mathematical induction to prove that $a_n = 2^n + 3^n$.

(d) Using the substitution $t = \tan\dfrac{x}{2}$, evaluate $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \operatorname{cosec} x\, dx$. **4**

(e) Find the vector equation of the line passing through points A an B, where $A = (2, -1, 3)$ and $B = (1, 4, 2)$. **2**

End of Question 13

Question 14 (15 marks) Begin a new Writing Booklet.

(a) An object moving in a straight line according to the equation

$$x = 6.5 + 5\sin 3t + 12\cos 3t.$$

where x is the displacement in metres and t is the time in seconds.

(i) Prove that the object is moving in simple harmonic motion by showing that x satisfies an equation of the form $\ddot{x} = -n^2(x - c)$. **2**

(ii) When is the displacement of the object zero for the first time? **3**

(b) $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ are two lines with vector equations:

$$\underset{\sim}{r}_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\underset{\sim}{r}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

where $\lambda, \mu \in \mathbb{R}$.

(i) Show that these two lines intersect. **2**

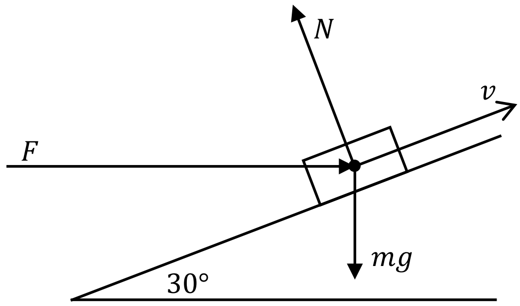
(ii) Find the angle between the lines. **1**

(iii) Find the shortest distance from the point $P(1, 2, 2)$ to the line **3**

$$\underset{\sim}{r}_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

(c) An object of 20 kg mass moves on a smooth inclined plane with velocity, $v(t)$, where t is time in seconds. The inclined plane makes an angle of 30° with the horizontal and the normal reaction force is N, as shown in the diagram.

The acceleration due to gravity is 9.8 ms^{-2}, whilst a lateral constant force F is applied in the direction of the upward slope.



i. Show that the resultant force on the object, parallel to the slope of the plane, is **2**

$$R = \frac{1}{2}\left(\sqrt{3}F - 196\right).$$

ii. If the object is initially at rest, determine the minimum lateral force, in Newtons (kg ms^{-2}), required to push the object 1 m up the slope within 1 minute. **2**

End of Question 14

Question 15 (15 marks) Begin a new Writing Booklet.

(a) A plastic toy is released on the surface of the ocean, at which point it immediately sinks. Let gravity be $10\ \text{ms}^{-2}$ and the resistance due to the water is proportional to the square of the velocity.

(i) Explain why the acceleration of the stone can be given by **1**

$$a = 10 - \frac{k}{m}v^2.$$

(ii) Given $\frac{k}{m} = \frac{1}{10}$, show that $v = 10\left(\frac{e^{2t}-1}{e^{2t}+1}\right)$. **4**

(iii) Using $a = v\frac{dv}{dx}$, show that $x = 5\ln\left(\frac{100}{100-v^2}\right)$. **2**

(iv) How far does the toy sink in the first 5 seconds? **2**

(b) Sketch the region of the Argand diagram that satisfies the following conditions **3**

$$|z-i| \le |z-2+i|$$
$$-\frac{\pi}{6} \le \text{Arg}(z) \le \frac{\pi}{6}$$

(c) Show that $x\sqrt{x} + 1 \ge x + \sqrt{x}$, for $x \ge 0$. **3**

End of Question 15

Question 16 (15 marks) Begin a new Writing Booklet.

(a)

i. Let $I_n = \int_1^e x(\ln x)^n dx$. **3**

Show that $I_n = \dfrac{e^2}{2} - \dfrac{n}{2} I_{n-1}, \qquad n = 1, 2, 3, \ldots$

ii. Hence evaluate I_2. **2**

(b) A model plane is launched from atop a tree house, 3 m above the ground with a velocity of $5\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}$ ms^{-1} where $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are all unit vectors in the east, north and vertically up directions, respectively.

The acceleration of the plane due to the combined effects of gravity and air resistance is $-2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$.

(i) Show that the angle above the horizontal at which the plane is launched is approximately 0.6 radians. **1**

(ii) Find the displacement vector for the plane. **3**

(iii) Find the maximum height reached by the model plane. **1**

(iv) Find the time taken for the plane to land. **1**

(v) At 3 seconds after launch, a sudden wind kicks up, impacting the flight of the plane, applying *additional* acceleration of $\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$. **4**

The new vector represents changes to the current system of forces acting upon the model plane.

How far from its intended destination will it now land?

End of Examination

Section I

Sunday, 4 August 2024 5:02 PM

HHHS
2024 Trial Exam
Mathematics Extension 2

Solutions and Marking Guidelines

1. $z = 3 - 2i$
$w = 1 + i$

$$\frac{z}{w} = \frac{3-2i}{1+i} \times \frac{1-i}{1-i}$$
$$= \frac{3 - 2i - 3i - 2}{1+1}$$
$$= \frac{1}{2} - \frac{5i}{2}$$

$\Rightarrow$ D

2. $\neg(P \Rightarrow (Q \Rightarrow R)) = P \wedge \neg(Q \Rightarrow R)$
$= P \wedge Q \wedge \neg R$

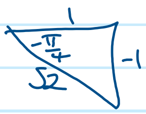
$\Rightarrow$ B

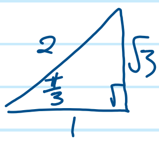
3. $(1+w)^2$ $\quad w \neq 1$
but $1 + w + w^2 = 0$
so $1 + w = -w^2$

$$(1+w)^2 = (-w^2)^2$$
$$= w^4$$
$$= w^3 \times w$$
$$= w$$

$\Rightarrow$ C

4. $\frac{1-i}{1+\sqrt{3}i}$





$$= \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right) \div 2\,\text{cis}\frac{\pi}{3}$$
$$= \frac{1}{\sqrt{2}}\,\text{cis}\left(-\frac{\pi}{4} - \frac{\pi}{3}\right)$$
$$= \frac{1}{\sqrt{2}}\,\text{cis}\left(-\frac{7\pi}{12}\right)$$

$\Rightarrow$ A

5. let $\underset{\sim}{a} = 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$.

A: $2(-2) + 1(2) - 1(-2) = -4 + 2 + 2$
$= 0 \quad \therefore$ perp.

but $|\underset{\sim}{A}| = \sqrt{(-2)^2 + 2^2 + (-2)^2}$
$= 2\sqrt{3} \neq 1$ so not a unit vector

B: $-1(2)+1(1)-1(-1) = -2+1+1$
$= 0$

$|\underset{\sim}{B}| = \frac{1}{\sqrt{3}}\sqrt{(-1)^2+1^2+(-1)^2}$

$= \frac{\sqrt{3}}{\sqrt{3}}$

$= 1$ $\Longrightarrow B$

6. $\int_{-a}^{a} f(x)dx = F(a) - F(-a)$

Consider $\int_0^a f(x)dx$

let $u = a-x \Rightarrow x = a-u$, $dx = -du$

$\int_0^a f(x)dx = -\int_0^a f(a-u)\,du$

$= \int_a^0 f(a-u)\,du$

$= \int_0^a (a-x)dx$

Consider $\int_{-a}^{0} f(x)dx$

Let $u = x+a$, $\Rightarrow x = u-a$, $dx = du$

$\int_{-a}^{0} f(x)dx = \int_0^a f(u-a)\,du$

$= \int_0^a f(x-a)\,dx$

$\therefore \int_{-a}^{a} f(x)dx = \int_0^a \left(f(x-a) + f(a-x)\right)dx$

$\Longrightarrow D$

7. $x = 6\sin 5t + 8\cos 5t.$

$v = 30\cos 5t - 40\sin 5t.$
$= 50\cos(5t+\varepsilon)$

amplitude is max velocity $\Longrightarrow D$

8. $\overrightarrow{AB} = k\overrightarrow{CB}$

$$\begin{pmatrix} -2-a \\ b-2 \\ 1-(-1) \end{pmatrix} = \begin{pmatrix} 3+2 \\ -1-b \\ 4-1 \end{pmatrix} k \qquad \begin{array}{l} -i \\ -ii \\ -iii \end{array}$$

from (iii) $k = \frac{2}{3}$

④ $b-2=\frac{2}{3}(-1-b)$

$3b-6=-2-2b$

$5b=4$

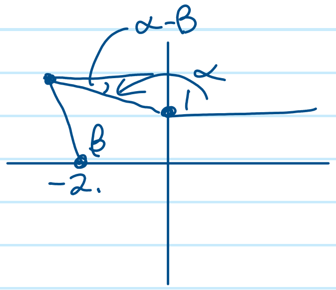
$b=\frac{4}{5}$

① $-2-a=\frac{2}{3}(3+2)$

$-a=\frac{10}{3}+2$

$a=-\frac{16}{3}$ $\Rightarrow$ C

9.



$\Rightarrow$ A

10. $a=8-v^2$

$v\frac{dv}{dx}=8-v^2$

$-\frac{1}{2}\int_2^{\sqrt{3}}\frac{-2v\,dv}{8-v^2}=\int_0^x dx$

$-\frac{1}{2}\left[\ln|8-v^2|\right]_2^{\sqrt{3}}=x$

$x=-\frac{1}{2}\left(\ln|8-3|-\ln|8-4|\right)$

$=-\frac{1}{2}\ln\left(\frac{5}{4}\right)$ $\Rightarrow$ B

Question 11

Sunday, 4 August 2024 5:02 PM

a) let $(x+iy)^2 = 15+8i$

$x^2 - y^2 + 2xyi = 15 + 8i$ — 1- squares $x+iy$

equating Re + Im

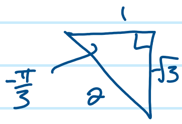
$x^2 - y^2 = 15$

$xy = 4$

by observation, $x=4$, $y=1$

$\therefore \sqrt{15+8i} = \pm(4+i)$ — 2- answers

b) i $1 - \sqrt{3}i = 2\,\text{cis}\left(-\frac{\pi}{3}\right)$



1- draws appr. △
— correct mod.
2- correct ans

ii $(1-\sqrt{3}i)^5 = \left[2\,\text{cis}\left(-\frac{\pi}{3}\right)\right]^5$

$= 2^5\,\text{cis}\left(-\frac{5\pi}{3}\right)$ — 1- applies de Moivre's

$= 32\,\text{cis}\,\frac{\pi}{3}$ — 2- expands — writes angle as $\frac{\pi}{3}$

$= 32\cos\frac{\pi}{3} + i.32\sin\frac{\pi}{3}$

$= 16 + 16\sqrt{3}\,i$ — 3 - correct ans

c). i $\int \sin^3 x\,dx = \int \sin^2 x \sin x\,dx$ — 1- splits $\sin^2.\sin$

$= \int (1-\cos^2 x)\sin x\,dx$

$= \int \left(\sin x + \cos^2 x(-\sin x)\right)dx$

$= -\cos x + \frac{\cos^3 x}{3} + C$ — 2- correct ans (ignore +c)

ii $\int \frac{x^3+2x+4}{x+1}\,dx$

$$\begin{array}{r|l} & x^2 - x + 3 \\ \hline x+1 & x^3 \quad\;\; + 2x + 4 \\ & x^3 + x^2 \\ \hline & -x^2 + 2x \\ & -x^2 - x \\ \hline & 3x + 4 \\ & 3x + 3 \\ \hline & 1 \end{array}$$

1- performs a step of polynomial div.

$\int \frac{x^3+2x+4}{x+1}dx$

$= \int \left(x^2 - x + 3 + \frac{1}{x+1}\right)dx$ — 2- ans.

$$= \frac{x^3}{3} - \frac{x^2}{2} + 3x + \ln|x+1| + C$$

d). let $\frac{7x+11}{(x-3)(x+1)^2} = \frac{A}{x-3} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

1 - correct partial fractions

$\therefore 7x+11 = A(x+1)^2 + B(x-3)(x+1) + C(x-3)$

for $x=3$

$7(3)+11 = A(3+1)^2$

$A = 2$

2 - finds a value.

for $x=-1$

$7(-1)+11 = C(-1-3)$

$C = -1$

for x^2 term

$0 = A + B$

$\therefore B = -2.$

$\therefore \frac{7x+11}{(x-3)(x+1)^2} = \frac{2}{x-3} - \frac{2}{x+1} - \frac{1}{(x+1)^2}$

3 - correct ans.

e). $z = 3e^{i\frac{\pi}{3}}$ $\quad w = 9e^{i\frac{\pi}{4}}$

$\frac{z}{w} = \frac{3e^{i\frac{\pi}{3}}}{9e^{i\frac{\pi}{4}}}$

1 - $e^{i(\frac{\pi}{3} - \frac{\pi}{4})}$

$= \frac{1}{3} e^{i(\frac{\pi}{3} - \frac{\pi}{4})}$

$= \frac{1}{3} e^{i\frac{\pi}{12}}$

2 - ans

Question 12

Sunday, 4 August 2024 5:02 PM

a) Assume $\sqrt{2+6n}$ is rational

$\therefore \sqrt{2+6n} = \frac{p}{q}$ where $p,q \in \mathbb{Z}$ and $GCD(p,q)=1$ | 1-opening statement

$2+6n = \frac{p^2}{q^2}$

$2q^2(1+3n) = p^2$ —i | 2-shows p even

p^2 is even $\Rightarrow p$ is even

so let $p = 2k$

$\therefore 2q^2(1+3n) = (2k)^2$

$q^2(1+3n) = 2k^2$

$2k^2$ is even $\Rightarrow q^2(1+3n)$ is even

as $1+3n$ may be even or odd,
q^2 must be even
$\Rightarrow q$ is even

and p and q have a common factor of 2. | 3-proof by contradict

$\therefore$ by contradiction, $\sqrt{2+6n}$ is irrational

b) i $P(z)$ has real coefficients, so complex roots exist in conjugate pairs

$2-3i$ is a zero $\Rightarrow 2+3i$ is a zero | 1- justifies conjugate z
let third root be α.

product of roots: $\frac{-(-26)}{1} = (2-3i)(2+3i)\alpha$

$\alpha = \frac{26}{2^2+3^2}$

$= 2$

$\therefore$ roots are
$2 \pm 3i,\ 2+0i$ | 2- correct roots

ii sum of pairs

$k = 2(2+3i) + 2(2-3i) + (2+3i)(2-3i)$ | 1 – correct value.

$= 8 + 4 + 9$

$= 21$

c). $I = \int_{\sqrt{7}}^{\sqrt{10}} x^3\sqrt{x^2-6}\,dx.$

let $u = x^2-6$ for $x=\sqrt{7}$, $u=1$

$du = 2x\,dx$ $x=\sqrt{10}$, $u=4$

1 - chooses and differentia[tes] appr. subst.

$I = \frac{1}{2}\int_1^4 (u+6)\sqrt{u}\,du$

2 - substitutes completely

$= \frac{1}{2}\left[\frac{2u^{\frac{5}{2}}}{5} + 6u^{\frac{3}{2}}\cdot\frac{2}{3}\right]_1^4$

$= \frac{1}{2}\left[\frac{2}{5}4^{\frac{5}{2}} + 4.\,4^{\frac{3}{2}} - \left(\frac{2}{5}(1)^{\frac{5}{2}} + 4.(1)^{\frac{3}{2}}\right)\right]$

$= \frac{1}{2}\left(\frac{7}{5}.32 - \frac{2}{5} - 4\right)$

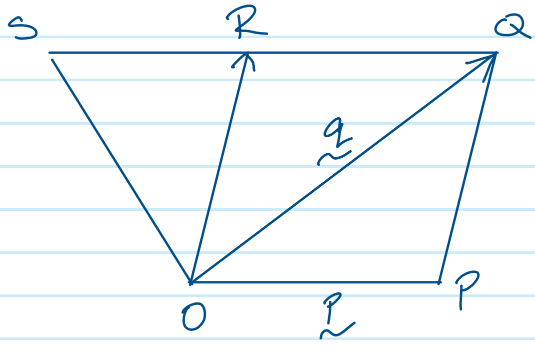
3 - correct ans

$= \frac{101}{5}$

d) i   an example is sufficient to prove a "there exists" statement

ii   a "for all" statement can be disproved by a counter-example.
e.g. Person B might have said it's not true for $x=2$

e).



$\vec{RQ} = \vec{OP} = \underset{\sim}{p}$

$\vec{SR} = \frac{2}{3}\vec{RQ}$

$= \frac{2}{3}\underset{\sim}{p}$

1 - notes relatio[n] between
- SR + R[Q]
- RQ + O[P]

$\vec{SQ} = \vec{SR} + \vec{RQ}$

2 - expression for SQ

$= \frac{2}{3}\underset{\sim}{p} + \underset{\sim}{p}$

$= \frac{5}{3}\underset{\sim}{p}$

3 - ans

$\vec{SP} = \vec{SQ} + \vec{QO} + \vec{OP}$

$= \frac{5}{3}\underset{\sim}{p} - \underset{\sim}{q} + \underset{\sim}{p}$

$= \frac{8}{3}\underset{\sim}{p} - \underset{\sim}{q}$

Question 13

Sunday, 4 August 2024 5:02 PM

a) let $z^5 = 0 + 32i$
$= 32e^{i\frac{\pi}{2}}$
$= 32e^{i(\frac{\pi}{2}+2k\pi)}$

$z = 2e^{i\left(\frac{\pi+4k\pi}{10}\right)}$, $k=0,1,2,3,4,\dots$
$= 2e^{i\frac{\pi}{10}}, 2e^{i\frac{5\pi}{10}}, 2e^{i\frac{9\pi}{10}}, 2e^{i\frac{13\pi}{10}}, 2e^{i\frac{17\pi}{10}}$
$= 2e^{i(-\frac{7\pi}{10})}, 2e^{i(-\frac{3\pi}{10})}, 2e^{i\frac{\pi}{10}}, 2e^{i\frac{5\pi}{10}}, 2e^{i\frac{9\pi}{10}}$

1 – writes in euler form or equiv.

2 – finds exp. for z

3 – correct ans

b) $\int \sin^3\theta\cos^4\theta \, d\theta = -\int (1-\cos^2\theta)\cos^4\theta(-\sin\theta)\,d\theta$
$= \int (\cos^6\theta - \cos^4\theta)(-\sin\theta)\,d\theta$
$= \frac{\cos^7\theta}{7} - \frac{\cos^5\theta}{5} + c$

1 – writes $\sin^2x$ as $1-\cos^2x$

2 – appr. intermediate form

3 – ans

c) $a_1 = 5, a_2 = 13 \quad a_n = 5a_{n-1} - 6a_{n-2} \quad n \geq 3$

Prove base cases

$a_1 = 2^1 + 3^1$
$= 5$

$a_2 = 2^2 + 3^2$
$= 4 + 9$
$= 13$

Therefore, statement true for $n=1$ and $n=2$

1 – addresses both base cases

Assume true for $n=3$ to $n=k$

$a_3 = 2^3 + 3^3$
$\vdots$
$\left.\begin{aligned} a_{k-1} &= 2^{k-1} + 3^{k-1} \\ a_k &= 2^k + 3^k \end{aligned}\right\} *$

2 – stronger assum. made ✓

Prove true for $n = k+1$
i.e. prove $a_{k+1} = 2^{k+1} + 3^{k+1}$

$a_{k+1} = 5a_k - 6a_{k-1}$
$= 5(2^k + 3^k) - 6(2^{k-1} + 3^{k-1})$ – by *

$= 5(2^k + 3^k) - 3.2^k - 2.3^k$

$= (5-3)2^k + (5-2).3^k$

$= 2^{k+1} + 3^{k+1}$ as required

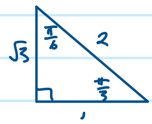
By the principal of induction, $a_n = 2^n + 3^n$ for all $n \geq 1$.

3- correct proof.

d) $t = \tan\frac{x}{2}$ $\quad x = \frac{\pi}{2}$ $t = \tan\frac{\pi}{4} = 1$

$x = 2\tan^{-1} t$ $\quad x = \frac{\pi}{3}$, $t = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

$dx = \frac{2dt}{1+t^2}$



1- finds dx in ter
of t
- change of bou

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \operatorname{cosec} x \, dx = \int_{\frac{1}{\sqrt{3}}}^{1} \frac{\cancel{1+t^2}}{\cancel{2}t} \cdot \frac{\cancel{2}dt}{\cancel{1+t^2}}$$

2- both the abov

3- correct subs

$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{dt}{t}$$

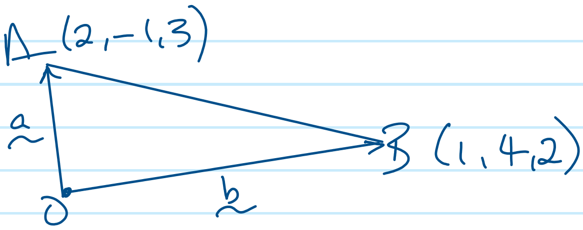
$$= \Big[\ln|t|\Big]_{\frac{1}{\sqrt{3}}}^{1}$$

$$= \ln 1 - \ln\frac{1}{\sqrt{3}}$$

$$= \frac{1}{2}\ln 3$$

4 - ans

e)



1- find b-a
or a-b

$\underset{\sim}{r} = \underset{\sim}{a} + \lambda(\underset{\sim}{b} - \underset{\sim}{a})$

$= \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1-2 \\ 4-(-1) \\ 2-3 \end{pmatrix}$

2-correct eq.

$= \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \underset{\pm}{+} \lambda \begin{pmatrix} -1 \\ 5 \\ -1 \end{pmatrix}$

also $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \overset{\pm}{+} \lambda \begin{pmatrix} -1 \\ 5 \\ -1 \end{pmatrix}$

Question 14

Sunday, 4 August 2024 5:02 PM

a) i

$x = 6.5 + 5\sin 3t + 12\cos 3t$

$\dot{x} = 15\cos 3t - 36\sin 3t$

$\ddot{x} = -45\sin 3t - 108\cos 3t$

$= -9(5\sin 3t + 12\cos 3t)$

$= -3^2(x - 6.5)$ as required

1 – differentiates once

2 – shown correctly

ii for $x = 0$

$6.5 + 5\sin 3t + 12\cos 3t = 0$

$5\sin 3t + 12\cos 3t = -6.5$

let $R\cos(3t - \alpha) = 5\sin 3t + 12\cos 3t$

$R\sin 3t\sin\alpha + R\cos 3t\cos\alpha = 5\sin 3t + 12\cos 3t$

$\therefore R\cos\alpha = 12$
$R\sin\alpha = 5$ $\Rightarrow R = 13$

1 – attempts to use auxilliary angle

$\alpha = \tan^{-1}\left(\frac{5}{12}\right)$

$\therefore 13\cos\left(3t - \tan^{-1}\left(\frac{5}{12}\right)\right) = -6.5$

$\cos\left(3t - \tan^{-1}\left(\frac{5}{12}\right)\right) = -\frac{1}{2}$

2 – correct form cos|sin

as $\cos\theta < 0$ θ in 2^{nd} & 3^{rd} quadrants

as $\cos\frac{\pi}{3} = \frac{1}{2}$

$0 \le 3t \le 6\pi$.

then $3t - \tan^{-1}\left(\frac{5}{12}\right) = \pi - \frac{\pi}{3}, \pi + \frac{\pi}{3}, \ldots$

$= \frac{2\pi}{3}, \frac{4\pi}{3}$

$3t = \frac{2\pi}{3} + 0.39479$

$3t = 2.48919$

$t = 0.82973$ seconds

3 – ans

b) i

let $\underset{\sim}{r_1} = \underset{\sim}{r_2}$

$$\begin{pmatrix}2\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}0\\-1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\-1\end{pmatrix} + \mu\begin{pmatrix}1\\2\\-1\end{pmatrix}$$

$\therefore \quad 2 + 0\lambda = 1 + \mu \quad \Rightarrow \mu = 1$

$\quad\;\; 0 - \lambda = 1 + \mu(2) \quad \Rightarrow \lambda = -3$

1- equates to find λ or μ

$1 + \lambda = -1 - \mu$

checking values

$LHS = 1 + (-3) \qquad RHS = -1 - 1$

$= -2 \qquad\qquad\qquad = -2$

$\qquad\qquad\qquad\qquad = LHS$

$\therefore$ lines intersect

2- confirms in 3rd equation

ii. compare direction vectors

$\underset{\sim}{v}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \qquad \underset{\sim}{v}_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

$$\cos\theta = \frac{\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2}{|\underset{\sim}{v}_1|\,|\underset{\sim}{v}_2|}$$

$$= \frac{0(1) - 1(2) + 1(-1)}{\sqrt{0^2 + (-1)^2 + 1^2}\,\sqrt{1^2 + 2^2 + (-1)^2}}$$

$$= \frac{-3}{\sqrt{2}\cdot\sqrt{6}}$$

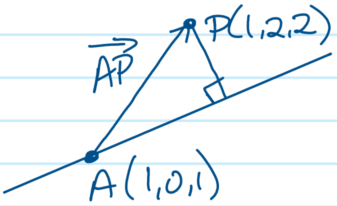
$$= -\frac{\sqrt{3}}{2}$$

$\theta = -0.523599$

$= 150^\circ \,/ 30^\circ$

1- correct angle

iii



$$\overrightarrow{AP} = \begin{pmatrix} 1-1 \\ 2-0 \\ 2-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

1- finds $\overrightarrow{AP}$

\- attempts pr

$$\text{Proj}_{\underset{\sim}{v}_1} \overrightarrow{AP} = \frac{\overrightarrow{AP} \cdot \underset{\sim}{v}_1}{\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_1}\, \underset{\sim}{v}_1$$

2- correct proj

$$= \frac{0(0) + 2(-1) + 1(1)}{0(0) - 1(-1) + 1(1)} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\text{Shortest distance} = \left| \overrightarrow{AP} - \text{Proj}_{\underset{\sim}{v}_1} \overrightarrow{AP} \right|$$

$$= \left| \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right|$$

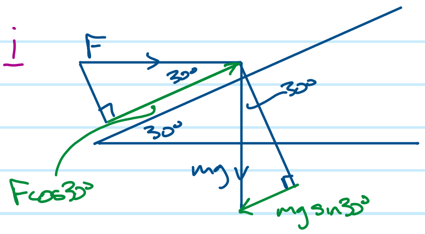
$$= \left| \begin{pmatrix} 0 \\ \frac{5}{2} \\ \frac{1}{2} \end{pmatrix} \right|$$

$$= \sqrt{0^2 + \left(\frac{5}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

3- ans

$$= \frac{\sqrt{26}}{2} \text{ units}$$
$$= 2.5495098$$

c) i



$$R = F\cos 30° - mg\sin 30°$$
$$= F\left(\frac{\sqrt{3}}{2}\right) - 20(9.8)\frac{1}{2}$$
$$= \frac{1}{2}\left(\sqrt{3}F - 196\right)$$

1 – appr. diagram
– finds one parallel component

2 – shown

ii

$$R = ma$$
$$\frac{1}{2}\left(\sqrt{3}F - 196\right) = 20a$$
$$a = \frac{1}{40}\left(\sqrt{3}F - 196\right)$$
$$\int_0^v dv = \frac{1}{40}\left(\sqrt{3}F - 196\right)\int_0^t dt$$
$$v = \frac{1}{40}\left(\sqrt{3}F - 196\right)\left[t\right]_0^t$$
$$= \frac{1}{40}\left(\sqrt{3}F - 196\right)t$$
$$\int_0^1 dx = \frac{1}{40}\left(\sqrt{3}F - 196\right)\int_0^{60} t\,dt$$
$$1 = \frac{1}{40}\left(\sqrt{3}F - 196\right)\left[\frac{t^2}{2}\right]_0^{60}$$
$$= \frac{3600}{80}\left(\sqrt{3}F - 196\right)$$
$$= 45\left(\sqrt{3}F - 196\right)$$
$$F = \frac{\frac{1}{45} + 196}{\sqrt{3}}$$
$$= 113.173493 \text{ N}$$

1 – correct exp for v

2 – ans.

Question 15

Sunday, 4 August 2024 5:02 PM

a) i

Given $F=ma$, the forces due to gravity and resistance are in opposite directions

1 - sound explanation noting opposing direction of forces

$$ma = F$$
$$= mg - kv^2.$$
$$a = g - \frac{k}{m}v^2$$

ii $\frac{k}{m} = \frac{1}{10}$

$$a = 10 - \frac{1}{10}v^2$$

$$\frac{dv}{dt} = \frac{1}{10}\left(100 - v^2\right)$$

1 - initial separation of variable.

$$\frac{dv}{100 - v^2} = \frac{1}{10}\,dt \qquad \frac{A}{10-v} + \frac{B}{10+v} = \frac{1}{100-v^2}$$

2 - correct partial fractions.

$$\frac{1}{20}\int_0^v \left(\frac{1}{10-v} + \frac{1}{10+v}\right) dv = \int_0^t \frac{1}{10}\,dt \qquad A(10+v) + B(10-v) = 1$$

$$v = 10 \Rightarrow A = \frac{1}{20}$$
$$v = -10 \Rightarrow B = \frac{1}{20}$$

$$\frac{1}{20}\Big[-\ln|10-v| + \ln|10+v|\Big]_0^v = \frac{1}{10}t$$

$$\ln\left|\frac{10+v}{10-v}\right| - \ln\left|\frac{10}{10}\right| = 2t$$

3 - integration

$$\ln\left|\frac{10+v}{10-v}\right| = 2t$$

$$\frac{10+v}{10-v} = e^{2t}$$

$$10 + v = e^{2t}(10 - v)$$

$$v\left(1 + e^{2t}\right) = 10\left(e^{2t} - 1\right)$$

$$\therefore \quad v = 10\,\frac{e^{2t}-1}{e^{2t}+1}$$

4 - result shown

iii.

$$a = 10 - \frac{1}{10}v^2$$

$$v\frac{dv}{dx} = \frac{1}{10}\left(100 - v^2\right)$$

$$-\frac{1}{2}\int_0^v \frac{-2v\,dv}{100 - v^2} = \frac{1}{10}\int_0^x dx$$

1 - integration after separation.

$$-\frac{1}{2}\Big[\ln|100 - v^2|\Big]_0^v = \frac{1}{10}\Big[x\Big]_0^x$$

$\frac{1}{2}\left(\ln 100 - \ln\left|100-v^2\right|\right) = \frac{1}{10}x$

$x = 5\ln\left(\frac{100}{100-v^2}\right)$ | 2 - shown

iv. $t=5$ $\quad v = 10\left(\frac{e^{2(5)}-1}{e^{2(5)}+1}\right)$ | 1 - finds v at t=...

$= 9.999092$

early rounding is problematic

$x = 5\ln\left(\frac{100}{100-9.999^2}\right)$

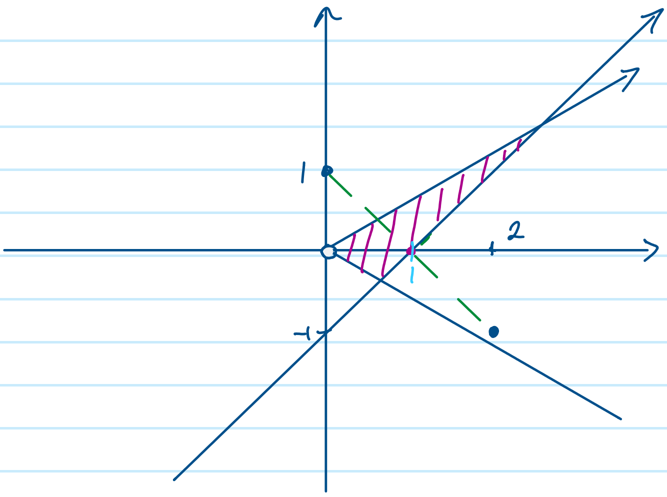
$= 43.0689482 \text{ m}$ | 2 - ans

b) $|z-i| \le |z-2+i|$

$|z-i| = |z-2+i|$ is the perpendicular bisector between $(0,1)$ and $(2,-1)$ | 1 - region between ... - perpendicular b...

$-\frac{\pi}{6} \le \text{Arg}(z) \le \frac{\pi}{6}$ region between two rays | 2 - region on li... - region with on... - correct region open circle

3 - correct con...



c) Show $x\sqrt{x}+1 > x+\sqrt{x}$, $x \ge 0$

$(LHS)^2 = (x\sqrt{x}+1)^2$ | 1 - squares LHS

$= x^3 + 2x\sqrt{x} + 1$

$= 2x\sqrt{x} + x^3+1$

$= 2x\sqrt{x} + (x+1)(x^2-x+1)$

$(x-1)^2 \ge 0$

$x^2-2x+1 \ge 0$ | 2 - uses inequality

$x^2+1 \ge 2x$

$(LHS)^2 \ge 2x\sqrt{x} + (x+1)(2x-x)$

$\ge 2x\sqrt{x} + x(x+1)$

$\ge x^2 + 2x\sqrt{x} + x$

$\ge (x+\sqrt{x})^2$

$\ge (RHS)^2$ | 3 - proof.

$\therefore LHS \ge RHS$ for $x \ge 0$

Question 16

Sunday, 4 August 2024 5:02 PM

a) i

$$I_n = \int_1^e x(\ln x)^n \, dx$$

let $u = (\ln x)^n$ $\quad dv = x\,dx$

$du = n(\ln x)^{n-1} \cdot \frac{1}{x} dx$ $\quad v = \frac{x^2}{2}$

1 – chooses u and dv correctly

$$I_n = \left[\frac{x^2}{2}(\ln x)^n\right]_1^e - n\int_1^e \frac{x}{2}(\ln x)^{n-1} dx$$

$$= \frac{e^2}{2}(\ln e)^n - \frac{1^2}{2}(\ln 1)^2 - \frac{n}{2} I_{n-1}$$

2 – uses integration by parts

$$= \frac{e^2}{2} - \frac{n}{2} I_{n-1}$$

3 – shown

ii

$$I_2 = \frac{e^2}{2} - \frac{2}{2} I_1$$

$$= \frac{e^2}{2} - I_1$$

1 – finds I_0 or I_2 in terms of I_1

$$I_1 = \frac{e^2}{2} - \frac{1}{2} I_0$$

$$I_0 = \int_1^e x\,dx$$

$$= \left[\frac{x^2}{2}\right]_1^e$$

$$= \frac{e^2}{2} - \frac{1}{2}$$

$$I_1 = \frac{e^2}{2} - \frac{1}{2}\left(\frac{e^2}{2} - \frac{1}{2}\right)$$

$$= \frac{e^2}{4} + \frac{1}{4}$$

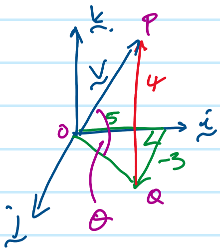
$$I_2 = \frac{e^2}{2} - I_1$$

$$= \frac{e^2}{2} - \left(\frac{e^2}{4} + \frac{1}{4}\right)$$

2 – correct value.

$$= \frac{e^2}{4} - \frac{1}{4}$$

b) i



$$|OQ| = \sqrt{5^2 + (-3)^2}$$

$$= \sqrt{34}$$

$$\theta = \tan^{-1}\frac{4}{\sqrt{34}}$$

$$= 0.60126.$$

1 – correctly shown inc. diagram(s)

ii $\underset{\sim}{a} = -2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$

$\underset{\sim}{v} = \int (-2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k})\, dt$

$= -2t\underset{\sim}{i} + t\underset{\sim}{j} - 2t\underset{\sim}{k} + c$

at $t=0$, $\underset{\sim}{v}_0 = 5\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k} = c$

1 – integrates

so $\underset{\sim}{v} = (5-2t)\underset{\sim}{i} + (-3+t)\underset{\sim}{j} + (4-2t)\underset{\sim}{k}$ 2 – finds $\underset{\sim}{v}$.

$\underset{\sim}{x} = \int ((5-2t)\underset{\sim}{i} + (-3+t)\underset{\sim}{j} + (4-2t)\underset{\sim}{k})\, dt$

$= (5t - t^2)\underset{\sim}{i} + (-3t + \frac{t^2}{2})\underset{\sim}{j} + (4t - t^2)\underset{\sim}{k} + c$

at $t=0$, $\underset{\sim}{x}_0 = 3\underset{\sim}{k} = c$

so $\underset{\sim}{x} = (5t - t^2)\underset{\sim}{i} + (-3t + \frac{t^2}{2})\underset{\sim}{j} + (3 + 4t - t^2)\underset{\sim}{k}$. 3 – finds $\underset{\sim}{x}$

iii max height when $\underset{\sim}{v}_k = 0$

$\therefore 4 - 2t = 0$

$t = 2$ s

at $t=2$, height $= 3 + 4(2) - (2)^2$

$= 7$ m 1 – correct ans

iv plane lands when $\underset{\sim}{x}_k = 0$

$\therefore 3 + 4t - t^2 = 0$

$t = \frac{-4 \pm \sqrt{4^2 - 4(3)(-1)}}{2(-1)}$

$= \frac{-4 \pm \sqrt{28}}{-2}$

$= 2 \pm \sqrt{7}$ 1 – correct ans

$= 4.6458$ s taking +

v at $t=3$

$\underset{\sim}{x}(3) = (5(3) - 3^2)\underset{\sim}{i} + (-3(3) + \frac{3^2}{2})\underset{\sim}{j} + (3 + 4(3) - 3^2)\underset{\sim}{k}$ 1 – finds $\underset{\sim}{x}(3)$

$= 6\underset{\sim}{i} - \frac{9}{2}\underset{\sim}{j} + 6\underset{\sim}{k}$ or $\underset{\sim}{v}(3)$

$\underset{\sim}{v}(3) = (5 - 2(3))\underset{\sim}{i} + (-3 + 3)\underset{\sim}{j} + (4 - 2(3))\underset{\sim}{k}$ – original dest.

$= -\underset{\sim}{i} + 0\underset{\sim}{j} - 2\underset{\sim}{k}$

new acceleration

$\underset{\sim}{a} = -2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k} + (\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k})$

$= -\underset{\sim}{i} + 4\underset{\sim}{j} - 3\underset{\sim}{k}$

$\underset{\sim}{v} = -t\underset{\sim}{i} + 4t\underset{\sim}{j} - 3t\underset{\sim}{k} + c$

$c = -\underset{\sim}{i} - 2\underset{\sim}{k}$

so $\underset{\sim}{v} = (-1-t)\underset{\sim}{i} + (4t)\underset{\sim}{j} + (-2-3t)\underset{\sim}{k}$

$\underset{\sim}{x} = \left(-t - \frac{t^2}{2}\right)\underset{\sim}{i} + (2t^2)\underset{\sim}{j} + \left(-2t - \frac{3t^2}{2}\right)\underset{\sim}{k} + c$

$c = 6\underset{\sim}{i} - \frac{9}{2}\underset{\sim}{j} + 6\underset{\sim}{k}$

so $\underset{\sim}{x} = \left(6 - t - \frac{t^2}{2}\right)\underset{\sim}{i} + \left(2t^2 - \frac{9}{2}\right)\underset{\sim}{j} + \left(6 - 2t - \frac{3t^2}{2}\right)\underset{\sim}{k}$.

2 – updated $\underset{\sim}{v}$,

– $\underset{\sim}{x}(3)$ or $\underset{\sim}{v}(3)$ + intend

Intended dest. at $t = 4.6458$

$(x, y) = (1.6458, -3.14575)$

3 – $\underset{\sim}{v}$ + intended

New time of flight: $6 - 2t - \frac{3t^2}{2} = 0$

$12 - 4t - 3t^2 = 0$

$t = 1.4415$

New dest. $(x, y) = (5.1296, -3.28557)$

distance from intended

$d = \sqrt{(5.1296 - 1.6458)^2 + (-3.28557 + 3.14575)^2}$

$= 3.4867\ m$

4 – ans