Student Number: ______________________

2013

TRIAL HSC EXAMINATION

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen. Black pen is preferred.
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- Show all necessary working in Questions 11–16.

Total Marks – 100

Section I Pages 2–3

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 4–10

90 marks

- Attempt Questions 11–16
- Allow about 2 hours 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 What is the supplement of $\frac{\pi}{6}$?

(A) $\frac{\pi}{3}$ (B) $\frac{5\pi}{6}$ (C) $\frac{5\pi}{3}$ (D) $\frac{11\pi}{6}$

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2 What is the equation of the parabola with vertex $(4, 2)$ and focus $(3, 2)$?

(A) $(x-4)^2 = 4(y-2)$ (B) $(x-4)^2 = -4(y-2)$

(C) $(y-2)^2 = 4(x-4)$ (D) $(y-2)^2 = -4(x-4)$

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3 The quadratic equation $2x^2 - 5x + 12 = 0$ has roots α and β. What is the value of $\alpha^2 + \beta^2$?

(A) $-\frac{23}{4}$ (B) $\frac{23}{4}$ (C) $\frac{25}{4}$ (D) $-\frac{25}{4}$

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4 The number represented by a 1 followed by one hundred zeros is called a googol. Which of the following is equal to a googol?

(A) 10^{10} (B) 10^{99} (C) 10^{100} (D) 10^{101}

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5 Which of the following expressions represents $\int \frac{x}{2x^2}\,dx$?

(A) $\frac{x}{2} + C$ (B) $\frac{1}{2}\ln x + C$ (C) $\ln\frac{x}{2} + C$ (D) $\frac{x^2}{4} + C$

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6 Given that $\sin\theta = \frac{5}{13}$ and $\tan\theta < 0$, what is the exact value of $\cos\theta$?

(A) $-\frac{12}{5}$ (B) $-\frac{12}{13}$ (C) $\frac{12}{13}$ (D) $\frac{12}{5}$

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7 Which of the following expressions is the correct simplification of $\dfrac{\operatorname{cosec}\theta \sec\theta}{\tan\theta}$?

(A) $\sin^2\theta$ (B) $\cos^2\theta$ (C) $\operatorname{cosec}^2\theta$ (D) $\sec^2\theta$

8 The curve $y = ax^2 - 6x + 3$ has a stationary point at $x = 1$. What is the value of a?

(A) 2 (B) –1 (C) 3 (D) –3

9 Which of the following correctly represents the sum $1 + x + x^2 + x^3 + \ldots + x^n$?

(A) $\sum_{k=1}^{n} x^k$ (B) $\sum_{k=1}^{n+1} x^k$ (C) $\sum_{k=1}^{n} x^{k-1}$ (D) $\sum_{k=1}^{n+1} x^{k-1}$

10 What is the range of the function $y = |x| - x$?

(A) All real y (B) $y \geq 0$ (C) $y \leq 0$ (D) $y = 0$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

All necessary working should be shown in every question.

Question 11 (15 marks) Use a new Writing Booklet

(a) Given that $n = 2-\sqrt{3}$, evaluate $n+\dfrac{1}{n}$, showing all working. Give your answer in exact form. **2**

(b) Differentiate the following with respect to x.

(i) $(x + 1)^2$ **2**

(ii) xe^{2x} **2**

(iii) $\ln \dfrac{x}{2x+1}$ **3**

(c) Write $\dfrac{x+1}{x(x-1)}-\dfrac{x-1}{x(x+1)}$ as a single fraction in simplest form. **3**

(d) Find $\int_0^1 1+e^{2x}\,dx$. Give your answer in exact form. **3**

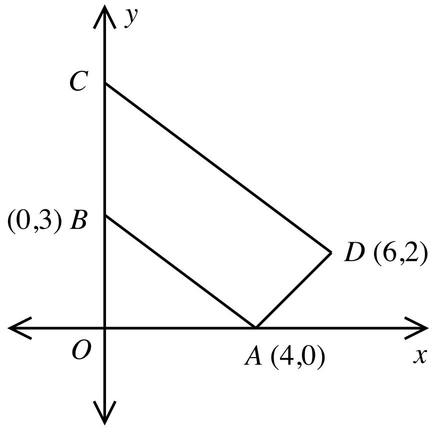
Question 12 (15 marks) Use a new Writing Booklet

(a) Using the trapezoidal rule, find an approximation to $\int_0^{\frac{\pi}{2}} \cos x \, dx$, using 4 function values. **3**

Give your answer correct to 3 significant figures.

(b) What is the domain and range of the curve $y = \ln(x + 1)$? **2**

(c)



The diagram above shows the points $A(4, 0)$, $B(0, 3)$ and $D(6, 2)$. The point C lies on the y-axis, and CD is parallel to AB.

(i) Copy of trace the diagram into your writing booklet.

(ii) What type of quadrilateral is $ABCD$? Give a reason to support your answer. **2**

(iii) Determine the gradient of the interval AB. **1**

(iv) Show that the equation of CD is $3x + 4y - 26 = 0$. **2**

(v) Find the coordinates of C. **1**

(vi) Show that the length of the interval AB is 5 units. **1**

(vii) Show that the length of the interval CD is $7\frac{1}{2}$ units. **1**

(viii) Find the perpendicular distance from A to CD. **1**

(ix) Hence, or otherwise, find the area of quadrilateral $ABCD$. **1**

Question 13 (15 marks) Use a new Writing Booklet

(a) For an arithmetic progression, the fifth term is 16 and the eleventh term is 40.

(i) Find the first term and the common difference. **3**

(ii) How many terms in the sequence must be added to reach a sum of 312? **2**

(b) Find the equation of the tangent to the curve $y = \log_e x^2$ at the point $(1, 0)$. **3**

(c) Consider the curve $y = 2 + 3x - x^3$.

(i) Find $\dfrac{dy}{dx}$. **1**

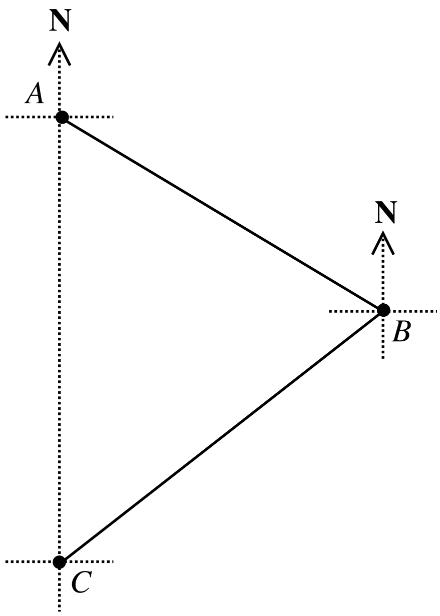
(ii) Locate any stationary points and determine their nature. **3**

(iii) For what value(s) of x is the curve concave up? **1**

(iv) Sketch the curve for $-2 \le x \le 2$. **2**

Question 14 (15 marks) Use a new Writing Booklet

(a) A bushwalker sets out from base camp (A) and walks for 14 km on a bearing of 121° to a lookout (B). She then turns and walks on a bearing of 232° until she is directly south of her starting point (C).



(i) Copy the diagram above into your writing booklet and complete it by clearly showing the above information. **1**

(ii) How far south is the bushwalker from her starting point? Give your answer correct to 1 decimal place. **3**

(b) (i) Write down the amplitude and period of the function $y = 2\cos 2x - 1$. **2**

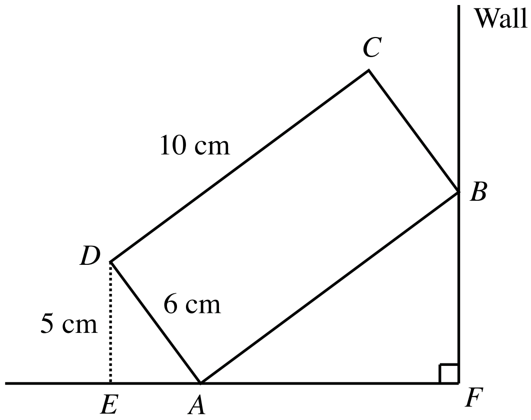
(ii) Draw a large, neat sketch of the function $y = 2\cos 2x - 1$ over the domain $-\pi \le x \le \pi$. **2**

(iii) On the same diagram, draw a neat sketch of the function $y = x - 1$. **1**

(iv) Hence find the number of solutions to the equation $2\cos 2x - x = 0$. **1**

Question 14 (continued)

(c)



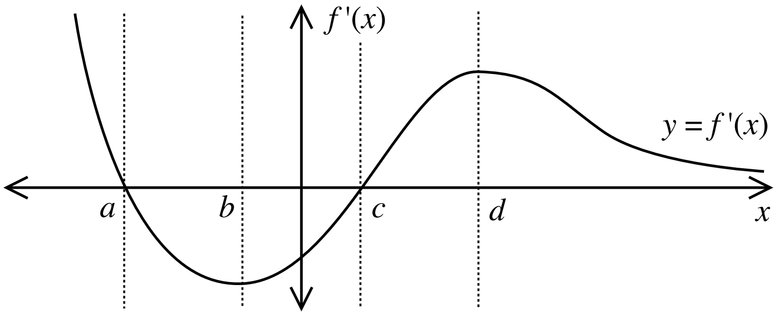
In the diagram above, $ABCD$ is the cross section of a rectangular block which has been placed against a wall. The block is 10 cm wide and 6 cm high. The outermost edge (indicated by the point D in the diagram) of the block is 5 cm above the ground.

(i) Prove that ΔEAD is similar to ΔFBA. **3**

(ii) At what height does the block touch the wall (indicated by the point B in the diagram)? Give your answer in simplest surd form. **2**

Question 15 (15 marks) Use a new Writing Booklet

(a) The diagram below shows the graph of a gradient function $y = f'(x)$.



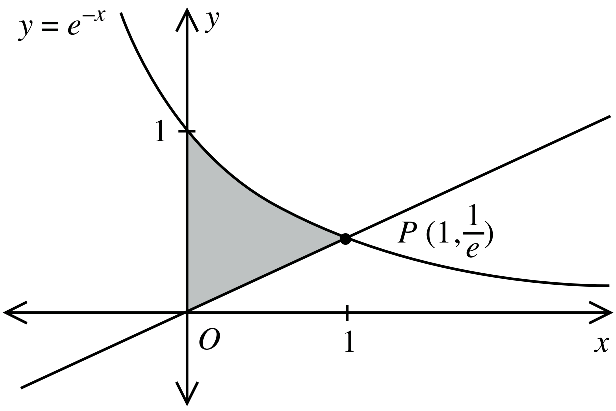
(i) Write down the value(s) of x where the curve is stationary. **2**

(ii) For what value of x will the curve have a maximum turning point? **1**

(iii) Copy or trace the diagram into your writing booklet. Draw a possible curve for $y = f(x)$, clearly showing what is happening to the curve as the values of x increase indefinitely. **2**

(b) Find the exact volume of the solid formed when the area enclosed by the curve $y = \dfrac{1}{\sqrt{x+1}}$ and the lines $x = 0$ to $x = 3$ is rotated about the x-axis. **4**

(c) The graph below shows the curve $y = e^{-x}$ intersecting with the line OP in the first quadrant. O is the origin and P has coordinates $(1, \frac{1}{e})$.



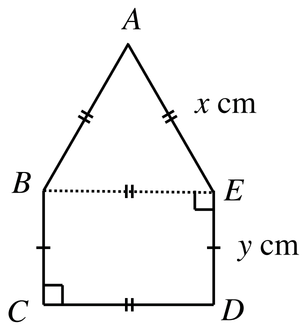
(i) Show that the line OP has equation $y = \dfrac{x}{e}$. **2**

(ii) Hence, or otherwise, calculate the shaded area bound by the curve $y = e^{-x}$, the line $y = \dfrac{x}{e}$ and the y-axis. Give your answer in exact form. **4**

Question 16 (15 marks) Use a new Writing Booklet

(a) Solve $2\sin^2\theta - \cos\theta + 1 = 0$ for the domain $0 \le \theta \le 2\pi$. **3**

(b) *ABCDE* is a pentagon with perimeter 30 cm. The pentagon is constructed with an equilateral triangle ΔABE joining a rectangle *BCDE*.



(i) Show that $y = \dfrac{30-3x}{2}$. **1**

(ii) Show that the area of ΔABE is $\dfrac{\sqrt{3}x^2}{4}$ cm². **2**

(iii) Hence show that the area of the pentagon is $15x + \dfrac{(\sqrt{3}-6)x^2}{4}$ cm². **2**

(iv) Find the exact value of x for which the area of the pentagon will be a maximum. Justify your solution. **3**

(c) Mr Jones has decided to start saving for an overseas trip when he takes long service leave in 5 years. He would like to have \$10 000 to cover all his expenses. He decides to save \$150 per month, at the start of each month, in an account which earns interest at 3% per annum, compounded monthly. Will Mr Jones reach his goal of \$10 000? By how much will he exceed or fall short of his goal? **4**

End of paper

2 UNIT MATHEMATICS
2013 TRIAL HSC EXAMINATION

SECTION I

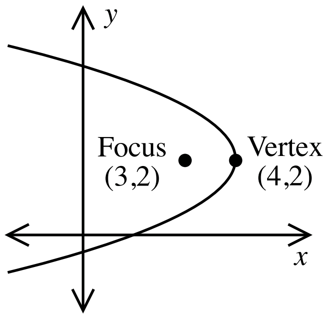
1 $\frac{\pi}{6} = 30°$ 1 **B**

$\therefore$ Supplement $= 180 - 30°$

$= 150°$

$= \frac{5\pi}{6}$

2 2 **D**



Focal length = 4 – 3 = 1 unit

Parabola is sideways, concave left.

$\therefore$ Equation is:

$(y - y_1)^2 = -4a(x - x_1)$

$(y-2)^2 = -4.1.(x-4)$

$(y-2)^2 = -4(x-4)$

3 $2x^2 - 5x + 12 = 0$ 3 **A**

$\alpha + \beta = \frac{-b}{a}$

$= \frac{-(-5)}{2}$

$= \frac{5}{2}$

$\alpha\beta = \frac{c}{a}$

$= \frac{12}{2}$

$= 6$

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= (\frac{5}{2})^2 - 2(6)$

$= -\frac{23}{4}$

4 $100 = 10^2$ 4 **C**

$1000 = 10^3$

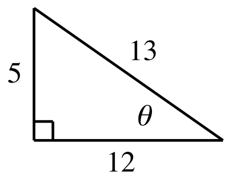
$\therefore$ 1 followed by 100 zeros $= 10^{100}$

5 $\int \frac{x}{2x^2}\,dx = \int \frac{1}{2x}\,dx$ 5 **B**

$= \frac{1}{2}\int \frac{1}{x}\,dx$

$= \frac{1}{2}\ \ln x + C$

6 6 **B**



If $\sin\theta > 0$ and $\tan\theta < 0$ then θ is in the second quadrant.
$\cos\theta = -\frac{12}{13}$

7 7 **C**

$$\frac{\operatorname{cosec}\theta \sec\theta}{\tan\theta} = \operatorname{cosec}\theta \times \sec\theta \div \tan\theta$$

$$= \frac{1}{\sin\theta} \times \frac{1}{\cos\theta} \div \frac{\sin\theta}{\cos\theta}$$

$$= \frac{1}{\sin\theta} \times \frac{1}{\cancel{\cos\theta}} \times \frac{\cancel{\cos\theta}}{\sin\theta}$$

$$= \frac{1}{\sin^2\theta}$$

$$= \operatorname{cosec}^2\theta$$

8 $y = ax^2 - 6x + 3$ 8 **C**

$\frac{dy}{dx} = 2ax - 6$

Stationary point at $x = 1$.

$\therefore$ When $x = 1$, $\frac{dy}{dx} = 0$.

Therefore:

$$2ax - 6 = 0$$
$$2a(1) - 6 = 0$$
$$2a - 6 = 0$$
$$2a = 6$$
$$a = 3$$

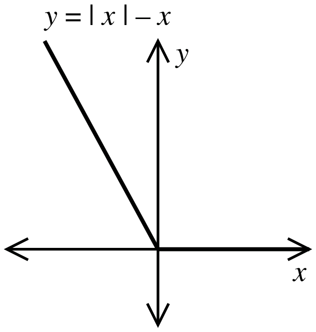
9 $1 + x + x^2 + x^3 + \ldots + x^n$ is a geometric progression with $n + 1$ terms. 9 **D**

When $k = 1$, $x^k = x^1$
$= x$

When $k = 1$, $x^{k-1} = x^{1-1}$
$= x^0$
$= 1$

$\therefore\ 1 + x + x^2 + x^3 + \ldots + x^n = \sum_{k=1}^{n+1} x^{k-1}$

10   10 **B**



When $x \geq 0$, $y = x - x$
$= 0$
The range of this section is $y = 0$.
When $x < 0$, $y = -x - x$
$= -2x$
The range of this section is $y > 0$.
$\therefore$ Range of function is $y \geq 0$.

SECTION II

QUESTION 11

(a)

$$\begin{aligned} n + \frac{1}{n} &= \frac{n^2+1}{n} \\ &= \frac{(2-\sqrt{3})^2+1}{2-\sqrt{3}} \\ &= \frac{4-4\sqrt{3}+3+1}{2-\sqrt{3}} \\ &= \frac{8-4\sqrt{3}}{2-\sqrt{3}} \\ &= \frac{4(2-\sqrt{3})}{2-\sqrt{3}} \\ &= 4 \end{aligned}$$

(b) (i) $y = (x+1)^2$

$$\begin{aligned} \frac{dy}{dx} &= 2.1.(x+1)^1 \\ &= 2(x+1) \\ &= 2x+2 \end{aligned}$$

(ii) $y = xe^{2x}$

$$\begin{aligned} \frac{dy}{dx} &= x.2e^{2x} + e^{2x}.1 \\ &= e^{2x}(2x+1) \end{aligned}$$

(iii) $y = \ln \frac{x}{2x+1}$

$= \ln x - \ln (2x + 1)$

$$\frac{dy}{dx} = \frac{1}{x} - \frac{2}{2x+1}$$

$$= \frac{2x+1}{x(2x+1)} - \frac{2x}{x(2x+1)}$$

$$= \frac{2x+1-2x}{x(2x+1)}$$

$$= \frac{1}{x(2x+1)}$$

or

$y = \ln \frac{x}{2x+1}$

Let $f(x) = \frac{x}{2x+1}$

$$f'(x) = \frac{(2x+1).1 - x(2)}{(2x+1)^2}$$

$$= \frac{2x+1-2x}{(2x+1)^2}$$

$$= \frac{1}{(2x+1)^2}$$

$$\therefore \frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

$$= \frac{1}{(2x+1)^2} \div \frac{x}{2x+1}$$

$$= \frac{1}{(2x+1)^2} \times \frac{2x+1}{x}$$

$$= \frac{1}{x(2x+1)}$$

(c)

$$\frac{x+1}{x(x-1)} - \frac{x-1}{x(x+1)} = \frac{(x+1)(x+1)}{x(x-1)(x+1)} - \frac{(x-1)(x-1)}{x(x+1)(x-1)}$$

$$= \frac{(x+1)(x+1)-(x-1)(x-1)}{x(x-1)(x+1)}$$

$$= \frac{(x^2+2x+1)-(x^2-2x+1)}{x(x-1)(x+1)}$$

$$= \frac{x^2+2x+1-x^2+2x-1}{x(x-1)(x+1)}$$

$$= \frac{4x}{x(x-1)(x+1)}$$

$$= \frac{4}{(x-1)(x+1)}$$

(d)

$$\int_0^1 1+e^{2x} = [x+\tfrac{1}{2}e^{2x}]_0^1$$
$$= [1+\tfrac{1}{2}e^2]-[0+\tfrac{1}{2}e^0]$$
$$= 1+\tfrac{1}{2}e^2-\tfrac{1}{2}$$
$$= \tfrac{1}{2}+\tfrac{1}{2}e^2$$
$$= \tfrac{1}{2}(1+e^2)$$
$$= \frac{1+e^2}{2}$$

QUESTION 12

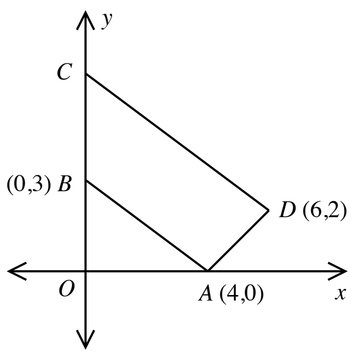
(a)

x	$f(x)$	W	P
0	1.0000	1	1.0000
$\frac{\pi}{6}$	0.8660	2	1.7321
$\frac{\pi}{3}$	0.5000	2	1.0000
$\frac{\pi}{2}$	0.0000	1	0.0000
			3.7321

$$\int_0^{\frac{\pi}{2}} \cos x \, dx \approx \tfrac{1}{2} \times h \times \text{Sum}$$
$$\approx \tfrac{1}{2} \times \tfrac{\pi}{6} \times 3.7321$$
$$\approx 0.9771$$

(b) Domain: $\{x: x > -1\}$
Range: $\{y: y \in \mathbb{R}\}$

(c) (i)



(ii) AB is parallel to CD (given)
$\therefore$ $ABCD$ is a trapezium (one pair of opposite sides parallel)

(iii) $m_{AB} = \dfrac{y_2 - y_1}{x_2 - x_1}$
$$= \frac{3-0}{0-4}$$
$$= -\tfrac{3}{4}$$

(iv) Since CD is parallel to AB, $m_{CD} = -\frac{3}{4}$

Equation of CD is:

$$y - y_1 = m(x - x_1)$$
$$y - 2 = -\tfrac{3}{4}(x - 6)$$
$$4(y-2) = -3(x-6)$$
$$4y - 8 = -3x + 18$$
$$3x + 4y - 26 = 0$$

(v) When $x = 0$:

$$3(0) + 4y - 26 = 0$$
$$4y - 26 = 0$$
$$4y = 26$$
$$y = 6\tfrac{1}{2}$$

$\therefore\ C \equiv (0, 6\tfrac{1}{2})$

(vi)

$$AB^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$
$$= (4-0)^2 + (0-3)^2$$
$$= 16 + 9$$
$$= 25$$
$$AB = 5 \text{ units}$$

(vii)

$$CD^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$
$$= (6-0)^2 + (2 - 6\tfrac{1}{2})^2$$
$$= 36 + 20\tfrac{1}{4}$$
$$= 56\tfrac{1}{4}$$
$$= \tfrac{225}{4}$$
$$CD = \tfrac{15}{2}$$
$$= 7\tfrac{1}{2} \text{ units}$$

(viii)

$$\text{Distance} = \left| \frac{ah + bk + c}{\sqrt{a^2 + b^2}} \right|$$
$$= \left| \frac{3.4 + 4.0 - 26}{\sqrt{3^2 + 4^2}} \right|$$
$$= \left| \frac{-14}{\sqrt{25}} \right|$$
$$= \frac{14}{5}$$
$$= 2\tfrac{4}{5} \text{ units}$$

(ix)

$$\text{Area } ABCD = \tfrac{1}{2}h(a+b)$$
$$= \tfrac{1}{2} \times 2\tfrac{4}{5} \times (5 + 7\tfrac{1}{2})$$
$$= 17\tfrac{1}{2} \text{ square units}$$

QUESTION 13

(a) (i) Solving simultaneously:

$$\begin{cases} T_5 = 16 \\ T_{11} = 40 \end{cases}$$

$$\begin{cases} a+(5-1)d = 16 \\ a+(11-1)d = 40 \end{cases}$$

$$\begin{cases} a+4d = 16 \\ a+10d = 40 \end{cases}$$

$$6d = 24$$
$$d = 4$$

Substituting:

$$a+4(4) = 16$$
$$a = 0$$

$\therefore$ First term = 0, common difference = 4

(ii) We need $S_n = 312$. Therefore:

$$S_n = \frac{n}{2}[2a+(n-1)d]$$
$$312 = \frac{n}{2}[2.0+(n-1).4]$$
$$312 = \frac{n}{2}[4n-4]$$
$$624 = n[4n-4]$$
$$624 = 4n^2 - 4n$$
$$4n^2 - 4n - 624 = 0$$
$$n^2 - n - 156 = 0$$
$$(n-13)(n+12) = 0$$

$\therefore$ n = 13 or –12

Since the term number must be positive we have n = 13.

$\therefore$ 13 terms in the sequence will add to a sum of 312.

(b) $y = \ln x^2$

$$\frac{dy}{dx} = \frac{2x}{x^2}$$
$$= \frac{2}{x}$$

When $x = 1$, $y = \ln 1^2$

$= \ln 1$

$= 0$

When $x = 1$, $\frac{dy}{dx} = \frac{2}{1}$

$= 2$

$\therefore$ Gradient of tangent to curve at (1,0) is 2.

$\therefore$ Equation of tangent is:

$$y - y_1 = m(x - x_1)$$
$$y - 0 = 2(x-1)$$
$$y = 2x - 2$$
$$2x - y - 2 = 0$$

(c) (i) $y = 2 + 3x - x^3$

$\frac{dy}{dx} = 3 - 3x^2$

$\frac{d^2y}{dx^2} = -6x$

(ii) For turning points, $\frac{dy}{dx} = 0$.

$3 - 3x^2 = 0$

$3(1 - x^2) = 0$

$3(1 + x)(1 - x) = 0$

$1 + x = 0$ or $1 - x = 0$

$x = -1$ or 1

When $x = -1$, $y = 2 + 3(-1) - (-1)^3$

$= 2 - 3 + 1$

$= 0$

When $x = -1$, $\frac{d^2y}{dx^2} = -6(-1)$

$= 6$

> 0

$\therefore$ Minimum turning point at $(-1,0)$

When $x = 1$, $y = 2 + 3(1) - (1)^3$

$= 2 + 3 - 1$

$= 4$

When $x = -1$, $\frac{d^2y}{dx^2} = -6(1)$

$= -6$

< 0

$\therefore$ Maximum turning point at $(1,4)$

For points of inflection, $\frac{d^2y}{dx^2} = 0$.

$-6x = 0$

$x = 0$

When $x = 0$, $y = 2 + 3(0) - (0)^3$

$= 2$

When $x = -0.1$, $\frac{d^2y}{dx^2} = -6(-0.1) > 0$

When $x = 0.1$, $\frac{d^2y}{dx^2} = -6(0.1) < 0$

$\therefore$ Point of inflection at $(0,2)$

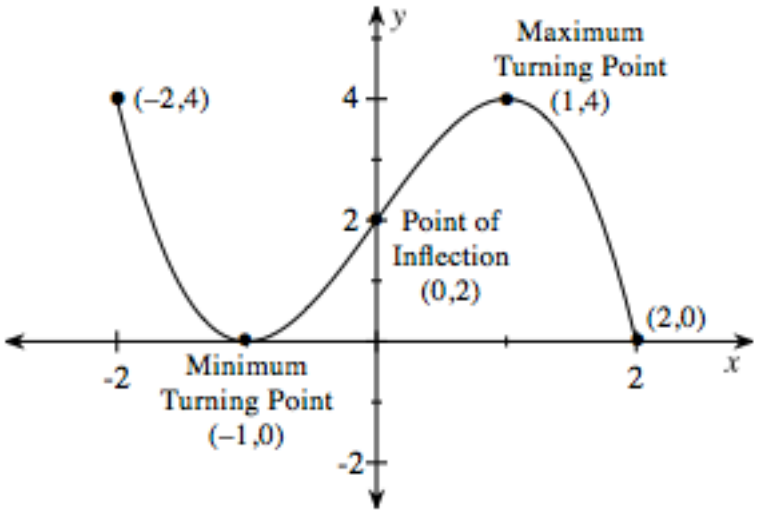
(iii) Curve is concave up when $\frac{d^2y}{dx^2} > 0$.

$-6x > 0$

$x < 0$

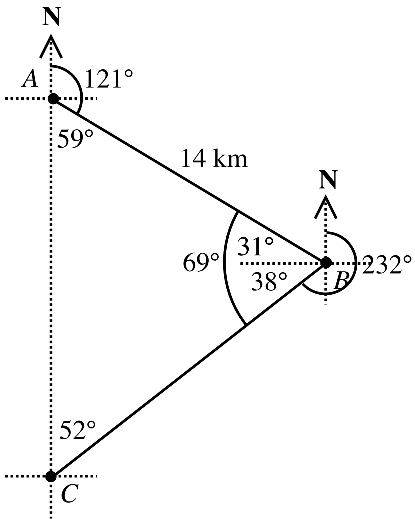
$\therefore$ Curve is concave up when $x < 0$.

(iv)



QUESTION 14

(a) (i)



(ii) Using the sine rule:

$$\frac{AC}{\sin 69°} = \frac{14}{\sin 52°}$$

$$AC = \frac{14\sin 69°}{\sin 52°}$$

$$= 16.58622793$$

$$= 16.6 \text{ km}$$

$\therefore$ The bushwalker is 16.6 km south from her starting point.

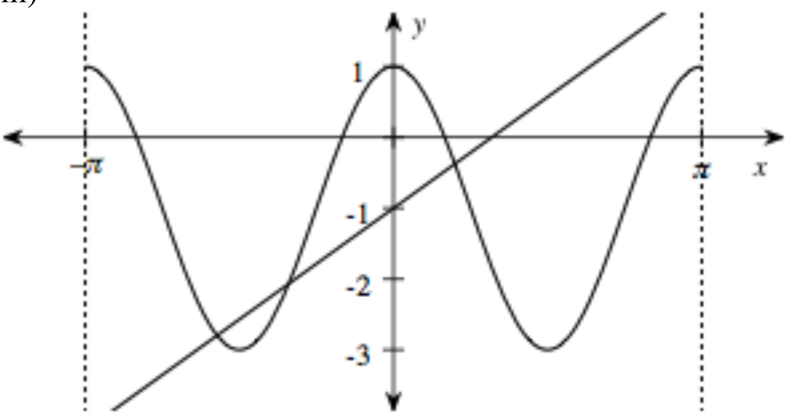
(b) (i) Amplitude = 2

Period $= \dfrac{2\pi}{n}$

$= \dfrac{2\pi}{2}$

$= \pi$

(ii), (iii)

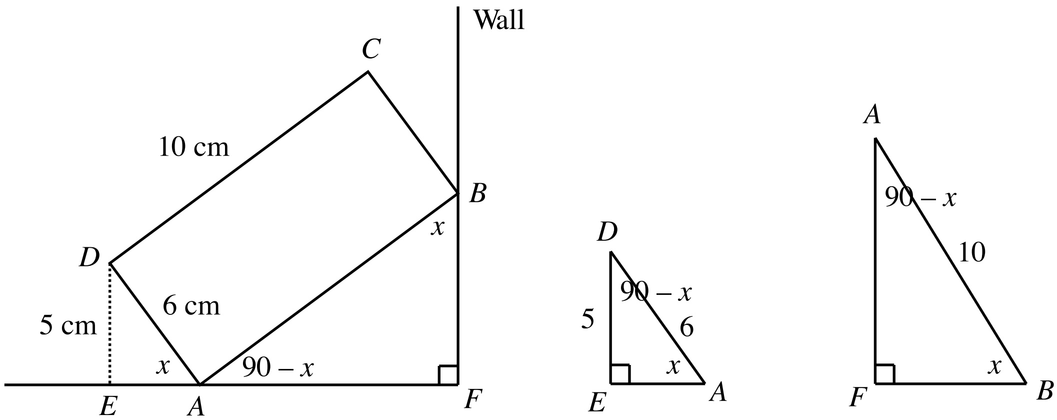


(iv) Solving simultaneously:

$$2\cos 2x - 1 = x - 1$$
$$2\cos 2x = x$$
$$2\cos 2x - x = 0$$

∴ There are 3 solutions to the equation $2\cos 2x - x = 0$.

(c)



(i) Let $x = \angle DAE$

$\angle BAF = 180 - 90 - x$ (supplementary angles)

$= 90 - x$

$\angle ABF = 180 - 90 - (90 - x)$ (angle sum of triangle)

$= 180 - 90 - 90 + x$

$= x$

∴ $\angle DAE = \angle ABF = x$ (as shown)

$\angle DEA = \angle AFB$ (given)

∴ ΔEAD is similar to ΔFBA (two pairs of corresponding angles equal)

(ii) Since the triangles are similar, we have:

$$\frac{AF}{DE} = \frac{AB}{DA}$$
$$\frac{AF}{5} = \frac{10}{6}$$
$$AF = \tfrac{25}{3}$$

Using Pythagoras' theorem:

$$\begin{aligned} BF^2 &= AB^2 - AF^2 \\ &= 10^2 - (\tfrac{25}{3})^2 \\ &= 100 - \tfrac{625}{9} \\ &= \tfrac{275}{9} \\ BF &= \sqrt{\tfrac{275}{9}} \\ &= \tfrac{\sqrt{275}}{3} \\ &= \tfrac{5\sqrt{11}}{3} \end{aligned}$$

$\therefore$ The block touches the wall at a height of $\tfrac{5\sqrt{11}}{3}$ cm above the floor.

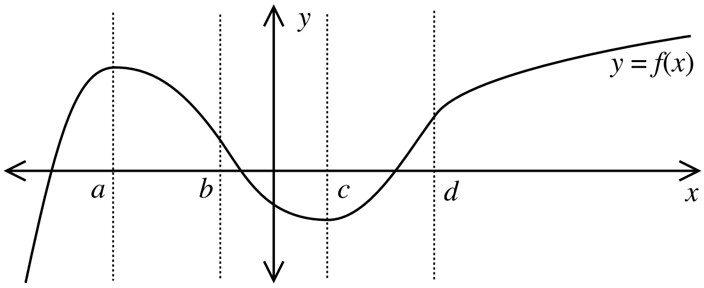
QUESTION 15

(a) (i) The curve is stationary when $x = b$ and $x = d$.

(ii) The curve has a maximum turning point when $x = d$.

(iii) Using the graph of $y = f'(x)$, we can see that $f'(x) = 0$ when $x = a$ and $x = c$.
Therefore turning points exist at $x = a$ and $x = c$.
Since $f'(x)$ changes from +ve to –ve at $x = a$, the turning point at $x = a$ is maximum.
Since $f'(x)$ changes from –ve to +ve at $x = c$, the turning point at $x = c$ is minimum.
Turning points of the graph $y = f'(x)$ indicate points of inflection on the graph of $y = f(x)$.
Therefore points of inflection exists at $x = b$ and $x = d$.
As $x \to \infty$, the gradient of $f(x)$ remains positive, but approaches zero.
This means the curve for $f(x)$ is increasing but flattens out.

A possible graph of $y = f(x)$ is shown.



(b) $$y = \frac{1}{\sqrt{x+1}}$$

$$y^2 = \left(\frac{1}{\sqrt{x+1}}\right)^2 = \frac{1}{x+1}$$

Therefore:

$$\begin{aligned}\text{Volume} &= \pi\int_0^3 y^2\,dx\\ &= \pi\int_0^3 \frac{1}{x+1}\,dx\\ &= \pi\left[\ln(x+1)\right]_0^3\\ &= \pi[\ln(3+1)]-\pi[\ln(0+1)]\\ &= \pi\ln 4-\pi\ln 1\\ &= \pi\ln 4-\pi\times 0\\ &= \pi\ln 4 \text{ cubic units}\end{aligned}$$

(c) (i) Equation of OP:

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$
$$\frac{y-0}{x-0}=\frac{\frac{1}{e}-0}{1-0}$$
$$\frac{y}{x}=\frac{\frac{1}{e}}{1}$$
$$\frac{y}{x}=\frac{1}{e}$$
$$ey=x$$
$$y=\frac{x}{e}$$

or

Gradient of OP $=\dfrac{y_2-y_1}{x_2-x_1}$

$$=\frac{\frac{1}{e}-0}{1-0}$$
$$=\frac{1}{e}$$

Equation of OP:

$$y-y_1=m(x-x_1)$$
$$y-0=\frac{1}{e}(x-0)$$
$$y=\frac{x}{e}$$

(ii) Let A be the point $(1,0)$.
Now in ΔPAO:

$$\begin{aligned}\text{Area} &= \tfrac{1}{2}bh\\ &= \tfrac{1}{2}\times 1\times\frac{1}{e}\\ &= \frac{1}{2e}\end{aligned}$$

or

$$\text{Area} = \int_0^1 \frac{x}{e}\,dx$$
$$= \frac{1}{e}\int_0^1 x\,dx$$
$$= \frac{1}{e}[\tfrac{1}{2}x^2]_0^1$$
$$= \frac{1}{e}[\tfrac{1}{2}(1)^2] - \frac{1}{e}[\tfrac{1}{2}(0)^2]$$
$$= \frac{1}{e}[\tfrac{1}{2}]$$
$$= \frac{1}{2e}$$

And under the curve $y = e^{-x}$:

$$\text{Area} = \int_0^1 e^{-x}\,dx$$
$$= \left[-e^{-x}\right]_0^1$$
$$= [-e^{-1}] - [-e^0]$$
$$= -\frac{1}{e} + 1$$
$$= 1 - \frac{1}{e}$$

Therefore:

$$\text{Shaded area} = \left(1 - \frac{1}{e}\right) - \frac{1}{2e}$$
$$= \frac{2e}{2e} - \frac{2}{2e} - \frac{1}{2e}$$
$$= \frac{2e-3}{2e} \text{ square units}$$

QUESTION 16

(a)

$$2\sin^2\theta - \cos\theta + 1 = 0$$
$$2(1-\cos^2\theta) - \cos\theta + 1 = 0$$
$$2 - 2\cos^2\theta - \cos\theta + 1 = 0$$
$$-2\cos^2\theta - \cos\theta + 3 = 0$$
$$2\cos^2\theta + \cos\theta - 3 = 0$$
$$(2\cos\theta + 3)(\cos\theta - 1) = 0$$

Therefore:

$$2\cos\theta + 3 = 0$$
$$2\cos\theta = -3$$
$$\cos\theta = -\tfrac{3}{2}$$

which has no solution.

or:

$$\cos\theta - 1 = 0$$
$$\cos\theta = 1$$
$$\theta = 0^\circ \text{ or } 360^\circ$$
$$= 0 \text{ or } 2\pi$$

$\therefore$ Solution to the equation is $\theta = 0$ or 2π.

(b) (i) Perimeter = 30 cm. Therefore:

$$x + y + x + y + x = 30$$
$$3x + 2y = 30$$
$$2y = 30 - 3x$$
$$y = \frac{30-3x}{2}$$
$$= 15 - \frac{3x}{2}$$

(ii) Using the area rule:

$$\text{Area of } \Delta ABE = \tfrac{1}{2} \times x \times x \times \sin 60^\circ$$
$$= \frac{x^2}{2} \times \frac{\sqrt{3}}{2}$$
$$= \frac{\sqrt{3}x^2}{4}$$

or

Using Pythagoras' theorem:

Let h be the perpendicular height of ΔABE. Therefore:

$$x^2 = h^2 + \left(\frac{x}{2}\right)^2$$
$$x^2 = h^2 + \frac{x^2}{4}$$
$$4x^2 = 4h^2 + x^2$$
$$3x^2 = 4h^2$$
$$4h^2 = 3x^2$$
$$h^2 = \frac{3x^2}{4}$$
$$h = \frac{\sqrt{3}x}{2}$$

Therefore:

$$\text{Area of } \Delta ABE = \tfrac{1}{2}bh$$
$$= \frac{1}{2} \times x \times \frac{\sqrt{3}x}{2}$$
$$= \frac{\sqrt{3}x^2}{4}$$

(iii)

$$\text{Area } BEDC = lb$$
$$= xy$$
$$= x \times \left(15 - \frac{3x}{2}\right)$$
$$= 15x - \frac{3x^2}{2}$$

$$\text{Area } AEDCB = \text{Area } \Delta AEB + \text{Area } BEDC$$
$$= \frac{\sqrt{3}x^2}{4} + 15x - \frac{3x^2}{2}$$
$$= \frac{\sqrt{3}x^2}{4} + \frac{60x}{4} - \frac{6x^2}{4}$$
$$= \frac{60x}{4} + \frac{\sqrt{3}x^2}{4} - \frac{6x^2}{4}$$
$$= 15x + \frac{(\sqrt{3}-6)x^2}{4}$$

(iv) $A = 15x + \dfrac{(\sqrt{3}-6)x^2}{4}$

$$\frac{dA}{dx} = 15 + \frac{(\sqrt{3}-6)x}{2}$$

$$\frac{d^2A}{dx^2} = \frac{\sqrt{3}-6}{2} < 0$$

For maximum area, we need $\frac{dA}{dx} = 0$ and $\frac{d^2A}{dx^2} < 0$. Therefore:

$$\begin{aligned}
15+\frac{(\sqrt{3}-6)x}{2} &= 0 \\
\frac{(\sqrt{3}-6)x}{2} &= -15 \\
(\sqrt{3}-6)x &= -30 \\
x &= \frac{-30}{\sqrt{3}-6} \\
&= \frac{30}{6-\sqrt{3}} \\
&= \frac{30}{6-\sqrt{3}} \times \frac{6+\sqrt{3}}{6+\sqrt{3}} \\
&= \frac{30(6+\sqrt{3})}{6^2-(\sqrt{3})^2} \\
&= \frac{30(6+\sqrt{3})}{36-3} \\
&= \frac{30(6+\sqrt{3})}{33} \\
&= \frac{10(6+\sqrt{3})}{11}
\end{aligned}$$

Since we know that $\frac{d^2A}{dx^2} < 0$, we have a maximum area when $x = \frac{10(6+\sqrt{3})}{11}$.

(c) Interest rate $= 3\%$ p.a.
$= 0.25\%$ per month
$= 0.0025$ per month

Let A_n = the final value of each amount invested.
The first \$150 is invested at 0.25% p.a. for 60 months.
The second \$150 is invested at 0.25% p.a. for 59 months.
The third \$150 is invested at 0.25% p.a. for 58 months.
The final \$150 is invested at 0.25% p.a. for 1 month.
Now:

$A_1 = 150(1 + 0.0025)^{60} = 150(1.0025)^{60}$
$A_2 = 150(1 + 0.0025)^{59} = 150(1.0025)^{59}$
$A_3 = 150(1 + 0.0025)^{58} = 150(1.0025)^{58}$
$A_{60} = 150(1 + 0.0025)^{1} = 150(1.0025)^{1}$

Therefore:

$$
\begin{aligned}
\text{Total} &= A_1 + A_2 + A_3 + \ldots + A_{60} \\
&= 150(1.0025)^{15} + 150(1.0025)^{14} + 150(1.0025)^{13} + \ldots + 150(1.0025)^{1} \\
&= 150(1.0025)^{1} + 150(1.0025)^{2} + 150(1.0025)^{3} + \ldots + 150(1.0025)^{15} \\
&= 150[\underbrace{1.0025^1 + 1.0025^2 + 1.0025^3 + \ldots + 1.0025^{60}}_{\text{GP: } a=1.0025,\, r=1.0025,\, n=60}] \\
&= 150\left[\frac{a(r^n - 1)}{r - 1}\right] \\
&= 150\left[\frac{1.0025(1.0025^{60} - 1)}{1.0025 - 1}\right] \\
&= 150 \times 64.8083294 \\
&= \$9721.249411 \\
&\approx \$9721.25
\end{aligned}
$$

$\therefore$ Mr Jones will not reach his goal and will fall short by \$278.75.