Proof of formula for partial fractions

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A general formula for partial fraction decomposition of rational functions

$$\frac{P(x)}{Q(x)} = p(x) + \sum_{i=1}^{m}\sum_{r=1}^{j_i} \frac{A_{ir}}{(x-a_i)^r} + \sum_{i=1}^{n}\sum_{r=1}^{k_i} \frac{B_{ir}x + C_{ir}}{(x^2+b_ix+c_i)^r}$$

was given in expanded form in Spivak's Calculus [1]. Yet he did not prove the formula. In this document we prove the formula.

The formula ultimately stems from the fundamental theorem of algebra introduced in but not proved in Sadler, et. at. [2] in section 3A as

Fundamental theorem of algebra. Every polynomial with degree > 1 has at least one zero, though that zero may be complex.

With lack of proof in the Sadler text we present a proof which actually proves it has n zeroes where $n = \deg(p(z))$.

Proof

If $p(z) := \sum_{j=0}^{n} a_n z^n$ with $a_n \neq 0$ and $R > 0$: $\left|\sum_{j=0}^{n-1} a_j z^j\right| \leq \sum_{j=0}^{n-1} |a_j| R^j < |a_n| R^n = |a_n z^n|$ for $|z| = R$

Rouché's theorem $\Rightarrow$ as $a_n z^n$ has n zeroes in $|z| < R$ then so too does $p(z)$ $\square$

There are many other proofs available such as in Fine and Rosenberger [3]

The Sadler text then goes on to prove a factorisation formula for a polynomial over $\mathbb{R}$ equivalent to $Q(x) = a\prod_{i=1}^{m}(x-a_i)^{j_i}\prod_{i=1}^{n}(x^2+b_ix+c_i)^{k_i}$ for $a, a_i, b_i, c_i \in \mathbb{R}$. This we will use later to prove the partial fractions formula.

But first we see in the Sadler text in section 6D what they call "A Theorem About Partial Fractions" which I rename to Lemma.

Lemma. Consider the rational function $\frac{P(x)}{A(x)\times B(x)}$, where P, A and B are polynomials, with no common factors between any pair, and where $\deg P < \deg A + \deg B$. It is always possible to write $\frac{P(x)}{A(x)\times B(x)} = \frac{R_A(x)}{A(x)} + \frac{R_B(x)}{B(x)}$, where the remainders R_A and R_B are polynomials with $\deg R_A < \deg A$ and $\deg R_B < \deg B$.

Again the Sadler text does not provide a proof of this Lemma and so again I include a proof.

Proof. By Bézout's identity for polynomials $\exists C(x), D(x) \in \mathbb{R}[x] : C(x)A(x) + D(x)B(x) = 1$

By the Euclidean division of $D(x)P(x)$ by $A(x)$ $\exists Q(x), R_A(x) \in \mathbb{R}[x]$ with $\deg R_A(x) < \deg A(x)$:

$D(x)P(x) = A(x)Q(x) + R_A(x)$ and so if $R_B(x) = C(x)P(x) + Q(x)B(x)$

$$\begin{aligned}
\therefore \frac{P(x)}{A(x)B(x)} &= \frac{P(x)(C(x)A(x)+D(x)B(x))}{A(x)B(x)} \\
&= \frac{D(x)P(x)}{A(x)} + \frac{C(x)P(x)}{B(x)} \\
&= \frac{R_A(x)+A(x)Q(x)}{A(x)} + \frac{R_B(x)-B(x)Q(x)}{B(x)} \\
&= \frac{R_A(x)}{A(x)} + Q(x) + \frac{R_B(x)}{B(x)} - Q(x) \\
&= \frac{R_A(x)}{A(x)} + \frac{R_B(x)}{B(x)}
\end{aligned}$$

It remains to show $\deg R_B(x) < \deg B(x)$. Since $P(x) = R_B(x)A(x) + R_A(x)B(x)$, then

$$\begin{aligned}
\deg R_B(x) &= \deg(P(x) - R_A(x)B(x)) - \deg A(x) \\
&\leq \max(\deg P(x), \deg(R_A(x)B(x))) - \deg A(x) \\
&< \deg(A(x)B(x)) - \deg A(x) \\
&= \deg B(x)
\end{aligned}$$

□

With repeated use of the Lemma we can now prove the partial fractions theorem.

Theorem. Given that $P(x)$ is a polynomial with real coefficients and $Q(x) = a\prod_{i=1}^{m}(x-a_i)^{j_i}\prod_{i=1}^{n}(x^2+b_ix+c_i)^{k_i}$ for $a, a_i, b_i, c_i \in \mathbb{R}$ where

- a_i are the m distinct real roots of $Q(x)$, each with a multiplicity of j_i.
- $x^2+b_ix+c_i$ are the n distinct irreducible quadratic factors of $Q(x)$ (meaning their discriminant is negative, $b_i^2 - 4c_i < 0$), each with a multiplicity of k_i.

Then

$$\frac{P(x)}{Q(x)} = p(x) + \sum_{i=1}^{m}\sum_{r=1}^{j_i}\frac{A_{ir}}{(x-a_i)^r} + \sum_{i=1}^{n}\sum_{r=1}^{k_i}\frac{B_{ir}x + C_{ir}}{(x^2+b_ix+c_i)^r}$$

where:

- $p(x)$ is the quotient polynomial obtained from polynomial long division if $\deg(P) \geq \deg(Q)$ (if $\deg(P) < \deg(Q)$, then $p(x) = 0$).
- A_{ir}, B_{ir} and C_{ir} are real constants.

Proof. If $\deg P \geq \deg Q$, perform polynomial long division: $P(x) = Q(x)p(x) + R(x)$, where $\deg R < \deg Q$.

Hence $\frac{P(x)}{Q(x)} = p(x) + \frac{R(x)}{Q(x)}$. Thus it suffices to decompose the proper rational function $\frac{R(x)}{Q(x)}$.

So from now on assume $\deg P < \deg Q$.

Let $Q(x) = A(x)B(x)$, where $A(x) = (x-a_1)^{j_1}, B(x) = \frac{Q(x)}{A(x)}$.

Since all factors of Q are distinct, $\gcd(A, B) = 1$.

The lemma therefore gives $\frac{P(x)}{Q(x)} = \frac{R_A(x)}{A(x)} + \frac{R_B(x)}{B(x)}$, where $\deg R_A < \deg A, \deg R_B < \deg B$.

Now apply the lemma again to the second term by splitting $B(x) = (x-a_2)^{j_2} \cdot \frac{Q(x)}{(x-a_1)^{j_1}(x-a_2)^{j_2}}$.

Continue this process. Each application separates off one relatively prime factor.

After finitely many steps we obtain

$$\frac{P(x)}{Q(x)} = \sum_{i=1}^{m} \frac{F_i(x)}{(x-a_i)^{j_i}} + \sum_{i=1}^{n} \frac{G_i(x)}{(x^2+b_ix+c_i)^{k_i}}, \text{ where } \deg F_i < j_i, \deg G_i < 2k_i ...(\dagger)$$

Consider $\frac{F(x)}{(x-a)^j}$ where $\deg F < j$. We prove by induction on j that $\frac{F(x)}{(x-a)^j} = \sum_{r=1}^{j} \frac{A_r}{(x-a)^r}$

For $j = 1$, F is constant.

Suppose it holds for $j-1$.

Since $\deg F < j$, $F(x) = F(a) + (x-a)H(x)$, where $\deg H < j-1$.

Hence $\frac{F(x)}{(x-a)^j} = \frac{F(a)}{(x-a)^j} + \frac{H(x)}{(x-a)^{j-1}}$, and the induction hypothesis applies to the second term.

Therefore $\frac{F_i(x)}{(x-a_i)^{j_i}} = \sum_{r=1}^{j_i} \frac{A_{ir}}{(x-a_i)^r}$.

Similarly consider $\frac{G(x)}{(x^2+bx+c)^k}$ for $\deg G < 2k$.

Since every polynomial can be divided by the quadratic, $G(x) = (Ux+V) + (x^2+bx+c)H(x)$, where $\deg H < 2(k-1)$.

Thus $\frac{G(x)}{(x^2+bx+c)^k} = \frac{Ux+V}{(x^2+bx+c)^k} + \frac{H(x)}{(x^2+bx+c)^{k-1}}$.

Applying induction on k, $\frac{G(x)}{(x^2+bx+c)^k} = \sum_{r=1}^{k} \frac{B_rx+C_r}{(x^2+bx+c)^r}$.

Hence $\frac{G_i(x)}{(x^2+b_ix+c_i)^{k_i}} = \sum_{r=1}^{k_i} \frac{B_{ir}x+C_{ir}}{(x^2+b_ix+c_i)^r}$.

Substituting the linear and quadratic decompositions into the expression ($\dagger$)

$\frac{P(x)}{Q(x)} = p(x) + \sum_{i=1}^{m} \sum_{r=1}^{j_i} \frac{A_{ir}}{(x-a_i)^r} + \sum_{i=1}^{n} \sum_{r=1}^{k_i} \frac{B_{ir}x+C_{ir}}{(x^2+b_ix+c_i)^r}$ where $A_{ir}, B_{ir}, C_{ir} \in \mathbb{R}$. $\square$

References

[1] Spivak, M., Calculus (4th ed.). Houston, Texas: Publish or Perish, 2008

[2] Sadler, D., Ward, D., McArthur, W., Mathematics Extension 2 CambridgeMaths NSW, 2nd ed, CUP, 2026

[3] Fine, B., and Rosenberger, G., The Fundamental Theorem of Algebra. Undergraduate Texts in Mathematics. New York: Springer, 1997