Student Number:

Teacher:

St George Girls High School

Mathematics Extension 2

2024 Trial HSC Examination

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in **Section I**, use the Multiple-Choice answer sheet provided

  For questions in **Section II:**
  - Answer the questions in the booklets provided
  - Start each question in a new writing booklet
  - Show relevant mathematical reasoning and/or calculations
  - Marks may not be awarded for incomplete or poorly presented solutions, or where multiple solutions are provided

Total marks: 100

Section I – 10 marks (pages 3 – 7)

- Attempt Questions 1– 10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 8 –13)

- Attempt Questions 11–16
- Allow about 2 hour and 45 minutes for this section

Q1-10	/10
Q11	/16
Q12	/16
Q13	/15
Q14	/16
Q15	/14
Q16	/13
TOTAL	**/100**
	%

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section
Use the multiple-choice answer sheet provided for Questions 1 to 10.

1. In which quadrant does the number $3e^{-\frac{i\pi}{3}} + 3e^{\frac{i\pi}{6}}$ lie?

A. I
B. II
C. III
D. IV

2. Consider the following statement:

"If Mollie O'Callaghan attends all training sessions, then she will win a gold medal at the Paris Olympic Games."

Which of the following is the negation of the statement?

A. Mollie O'Callaghan does not attend all training sessions, and she does win a gold medal at the Paris Olympic Games.

B. Mollie O'Callaghan attends all training sessions, and she does not win a gold medal at the Paris Olympic Games.

C. Mollie O'Callaghan does not attend all training sessions, and she does not win a gold medal at the Paris Olympic Games.

D. If a gold medal was won at the Olympic Games, then Mollie O'Callaghan attended all training sessions.

3. For $z \in C$, if $\text{Im}(z) > 0$, then $\arg\left(\frac{z\bar{z}}{z-\bar{z}}\right)$ is :

A. $-\frac{\pi}{2}$

B. 0

C. $\frac{\pi}{4}$

D. $\frac{\pi}{2}$

4. Which expression is equal to $\int \frac{dx}{\sqrt{8-2x-x^2}}$?

A. $\sin^{-1}\left(\frac{1-x}{2\sqrt{2}}\right)+C$

B. $\sin^{-1}\left(\frac{1-x}{3}\right)+C$

C. $\sin^{-1}\left(\frac{1+x}{2\sqrt{2}}\right)+C$

D. $\sin^{-1}\left(\frac{1+x}{3}\right)+C$

5. Consider the two statements:

P : $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $y^3 = x$.

Q : $\exists y \in \mathbb{R}$, such that $\forall x \in \mathbb{R}, y^3 = x$.

Which statement best represents the truth of each of P and Q?

A. P is true and Q is true.

B. P is true and Q is false.

C. P is false and Q is true.

D. P is false and Q is false.

6. Which of the following integrals is equal to zero?

A. $\int_0^a x^3(e^x+e^{-x})dx$

B. $\int_{-a}^a x^3(e^x+e^{-x})dx$

C. $\int_0^a \frac{x^3}{(e^x+e^{-x})}dx$

D. $\int_{-a}^a \frac{\cos x}{(e^x+e^{-x})}dx$

7. Consider the complex numbers $z_1 = (ln\alpha)^3 + i(ln\alpha)^2$ and $z_2 = -2ln\alpha + 8i$.
When z_2 is rotated about the origin by $\frac{\pi}{2}$ in an anti-clockwise direction, we get z_3.
For which value(s) of α do z_1 and z_3 coincide?

A. $\alpha = 1$

B. $\alpha = e^{-2}$

C. $\alpha = 1\ or\ \alpha = e^{-2}$

D. $\alpha = 0\ or\ \alpha = e^{-2}$

8. Using the substitution $u = 1 + e^x$, $\int_0^{log_e 2} \frac{1}{1+e^x}\, dx$ can be expressed as :

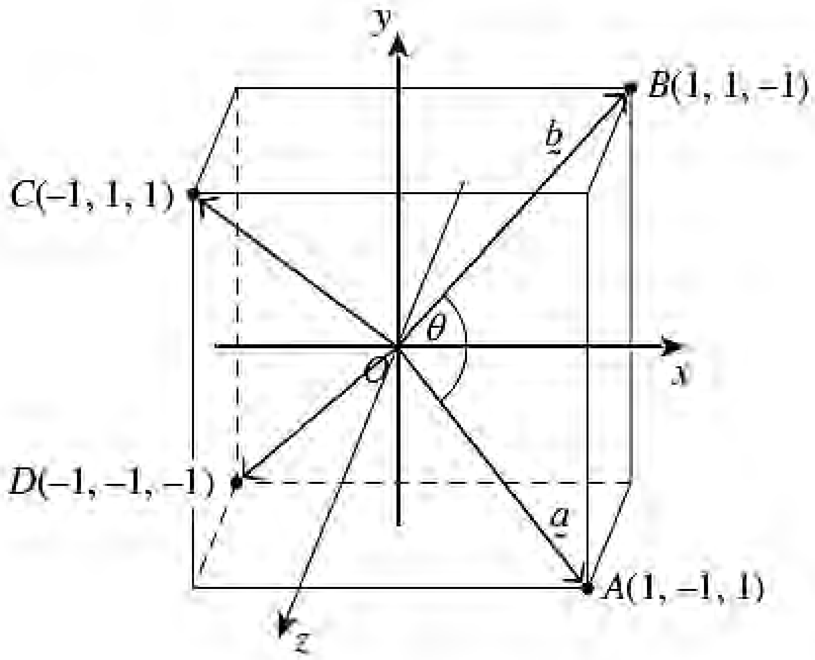
A. $\int_0^{log_e 2} \left(\frac{1}{u-1} - \frac{1}{u}\right) du$

B. $\int_2^{3} \left(\frac{1}{u} - \frac{1}{u-1}\right) du$

C. $\int_1^{3} \left(\frac{1}{u} - \frac{1}{u-1}\right) du$

D. $\int_2^{3} \left(\frac{1}{u-1} - \frac{1}{u}\right) du$

9. Consider the diagram below.



What is the size of $\angle BOA$ correct to one decimal place.

A. 54.7°

B. 70.5°

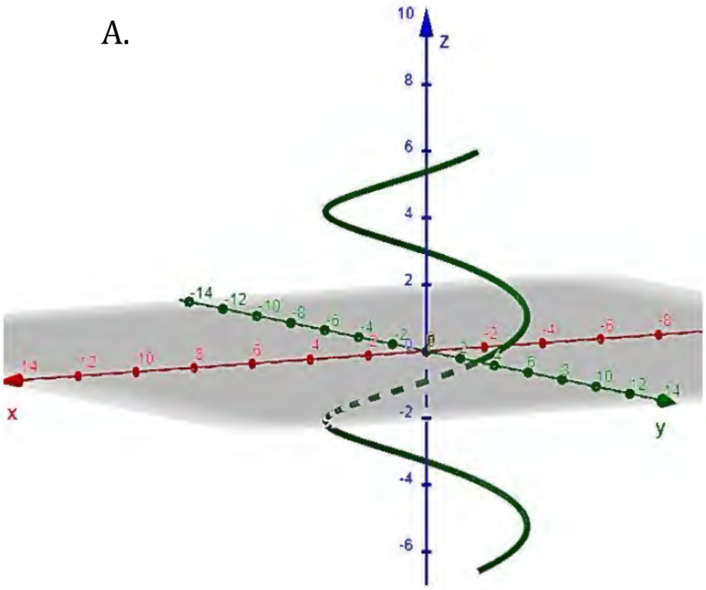
C. 109.5°

D. 179.0°

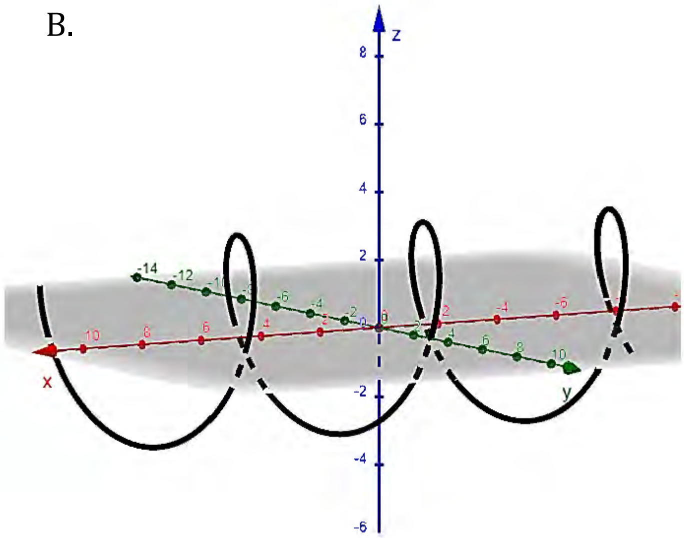
10. Which diagram best shows the curve described by the position vector

$$\underset{\sim}{r}(t) = -3\sin(t)\underset{\sim}{i} + 3\cos(t)\underset{\sim}{j} + t\underset{\sim}{k} \text{ for } -2\pi \leq t \leq 2\pi\,?$$

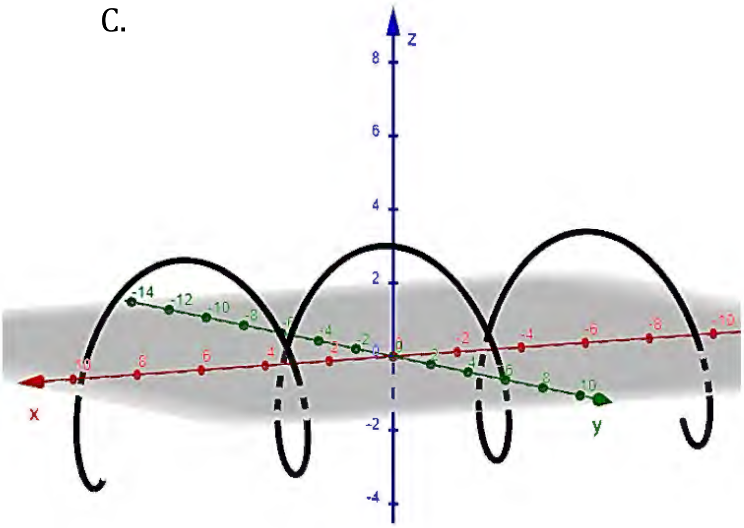
A.



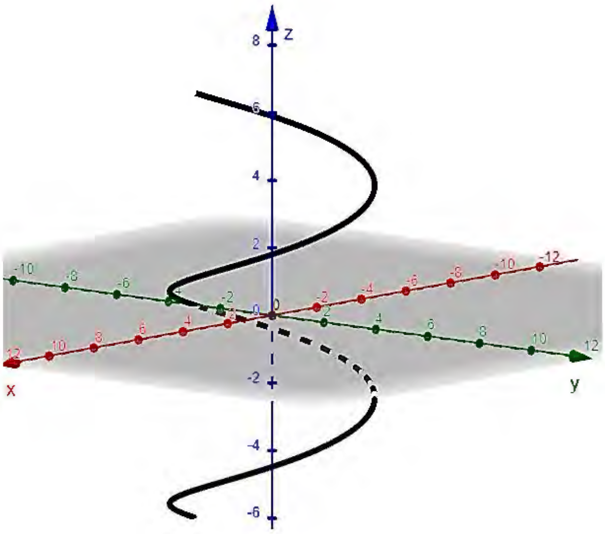
B.



C.



D.



END OF SECTION I

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section

In Questions 11-16, your responses should include relevant mathematical reasoning and/or calculations

Question 11 (16 marks) START A NEW WRITING BOOKLET. **Marks**

(a) Let $z = 3 - 2i$ and $\omega = -1 - i$. Find:

(i) $z\overline{\omega}$ in cartesian form. 2

(ii) $\arg(\omega)$. 1

(iii) ω^{-12}. 2

(b) If $\overrightarrow{OA} = \underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}$ and $\overrightarrow{OB} = 3\underset{\sim}{i} - 3\underset{\sim}{j} - 2\underset{\sim}{k}$, find:

(i) the magnitude of $\overrightarrow{AB}$. 2

(ii) the position vector of length 14 that is in the same direction as $\overrightarrow{AB}$. 1

(iii) a vector equation of the line AB. 2

(c) Solve $z^2 - z + (4 - 2i) = 0$. Express your answer in the form $x + yi$,

where x and y are real. 3

(d) Evaluate $\displaystyle\int_0^{\pi} \sin^3 x \cos^2 x \, dx$. 3

Question 12 (16 marks) START A NEW WRITING BOOKLET. **Marks**

(a) The point $(2, y, z)$ lies on the line $\underset{\sim}{r} = \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -1 \\ 5 \end{pmatrix}$.

Find the values of y and z. **2**

(b) Find $\int \frac{x^3}{x^2+x+1}\, dx$. **3**

(c) The polynomial $p(z) = z^3 + az^2 + bz + c$, where $z \in C$ and $a, b, c \in \mathbb{R}$,

can also be written as $p(z) = (z - z_1)(z - z_2)(z - z_3)$, where $z_1 \in \mathbb{R}$

and $z_2, z_3 \notin \mathbb{R}$.

(i) State the relationship between z_2 and z_3 . **1**

(ii) Determine the values of a,b and c, given that $p(2) = -13$,

$|z_2 + z_3| = 0$ and $|z_2 - z_3| = 6$. **3**

(d) Prove or disprove the statement that for all $x \in \mathbb{R}$, $|2x + 1| \leq 5 \Rightarrow |x| \leq 2$. **2**

(e) Consider the following lines:

$$\underset{\sim}{r_1} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \underset{\sim}{r_2} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$$

(i) Prove that the lines $\underset{\sim}{r_1}$ and $\underset{\sim}{r_2}$ are not skew. **2**

(ii) Find the equation of a line that passes through the point of intersection of the two lines $\underset{\sim}{r_1}$ and $\underset{\sim}{r_2}$, that is also perpendicular to both lines. **3**

Question 13 (15 marks) START A NEW WRITING BOOKLET. **Marks**

(a) (i) Find the values for A, B and C such that $\frac{3x^2+4x+12}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$. 2

(ii) Hence, find $\int \frac{3x^2+4x+12}{x(x^2+4)}\,dx$. 2

(b) (i) What is the contrapositive of the statement :

"For $x \in Z$, if x^2+3x+1 is even, then x is odd." 1

(ii) Using the contrapositive, prove for $x \in Z$,
if x^2+3x+1 is even, then x is odd." 2

(c) Use integration by parts to find $\int e^{-x} \sin x \, dx$. 3

(d) Consider the point $z_4 = \sqrt{3} + i$.

(i) **Using the Argand diagram provided on pg15 of this exam,**

sketch the ray given by $\arg(z - z_4) = \frac{5\pi}{6}$. 2

The ray $\arg(z - z_4) = \frac{5\pi}{6}$ intersects the circle $|z - 3i| = 1$, dividing it into a major and a minor segment.

(ii) Sketch the circle $|z - 3i| = 1$ **on the same Argand diagram provided on pg15 of this exam.** 1

(iii) **In your answer booklet**, find the area of the minor segment. 2

Question 14 (16 marks) START A NEW WRITING BOOKLET. **Marks**

(a) (i) If $t = \tan\frac{x}{2}$, derive an expression for $\frac{dx}{dt}$ in terms of t. 1

(ii) Hence, find $\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x+\cos x}\,dx$. 2

(b) Prove that for all real numbers x and y, where $x \neq y$,

$$x^4 + y^4 + z^4 > x^2y^2 + x^2z^2 + z^2y^2.$$ 2

(c) S_1 is a sphere with centre $2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$ which passes through the origin.

S_2 is defined by the equation $x^2 + y^2 + z^2 - 12x - 12y - 16z + 100 = 0$.

Do the spheres touch each other OR do they intersect. Justify your answer with appropriate mathematical calculations. 4

(d) Prove by contradiction that for $a \geq 2$, $\sqrt{a} + \sqrt{a+2} > \sqrt{a+8}$. 3

(e) Evaluate $\int_1^{\sqrt{3}} \frac{1}{x^2\sqrt{1+x^2}}\,dx$. 4

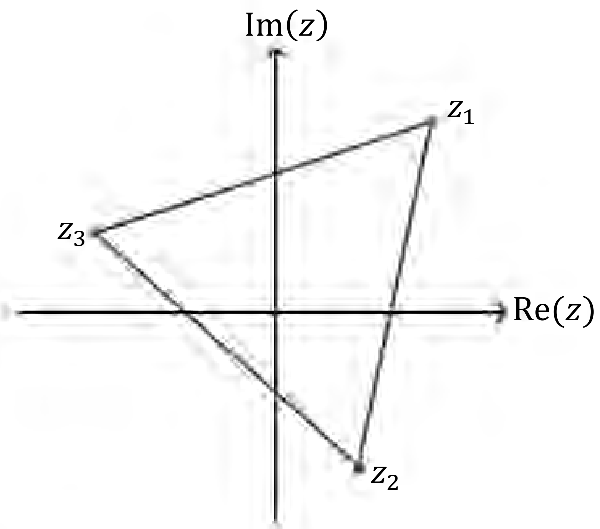
Question 15 (14 marks) START A NEW WRITING BOOKLET. **Marks**

(a) (i) Show $\frac{2k}{k+2} < \frac{2k+2}{k+3}$ for $k > 0$. 2

(ii) Use mathematical induction to prove that :

$\frac{1}{3!} + \frac{2}{4!} + \frac{3}{5!} + \cdots + \frac{n}{(n+2)!} < \frac{2n}{n+2} - \frac{1}{(n+2)!}$ for all integers $n \geq 1$. 3

(b) Let z_1, z_2 and z_3 form an equilateral triangle as shown below.



(i) Show that $\frac{z_2 - z_1}{z_3 - z_1} = \text{cis}\left(\frac{\pi}{3}\right)$. 2

(ii) Deduce that $z_1{}^2 + z_2{}^2 + z_3{}^2 = z_1 z_3 + z_1 z_2 + z_2 z_3$. 2

(c) (i) Prove that $\int_{-a}^{a} f(x)dx = \int_{0}^{a} [f(x) + f(-x)]dx$. 2

(ii) Use the identity from part (i) to calculate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^x \sin^2 x}{1 + e^x} dx$. 3

Question 16 (13 marks) START A NEW WRITING BOOKLET. **Marks**

(a) A sequence is defined by the following formula for $n \in \mathbb{Z}^+$:

$$T_0 = 0$$

$$T_n = \sqrt{T_{n-1} + 2}$$

Prove by mathematical induction that $T_n = 2\cos\left(\frac{\pi}{2^{n+1}}\right)$ for integer $n \geq 0$. 4

(b) Consider $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx$, $n \geq 0$.

(i) For $n \geq 2$, show that $I_n = \frac{n-1}{n} I_{n-2}$. 3

(ii) Hence, show that $I_{2n} = \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times \cdots \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$ and

$$I_{2n+1} = \frac{2n}{2n+1} \times \frac{2n-2}{2n-1} \times \cdots \times \frac{4}{5} \times \frac{2}{3} \times 1.$$ 4

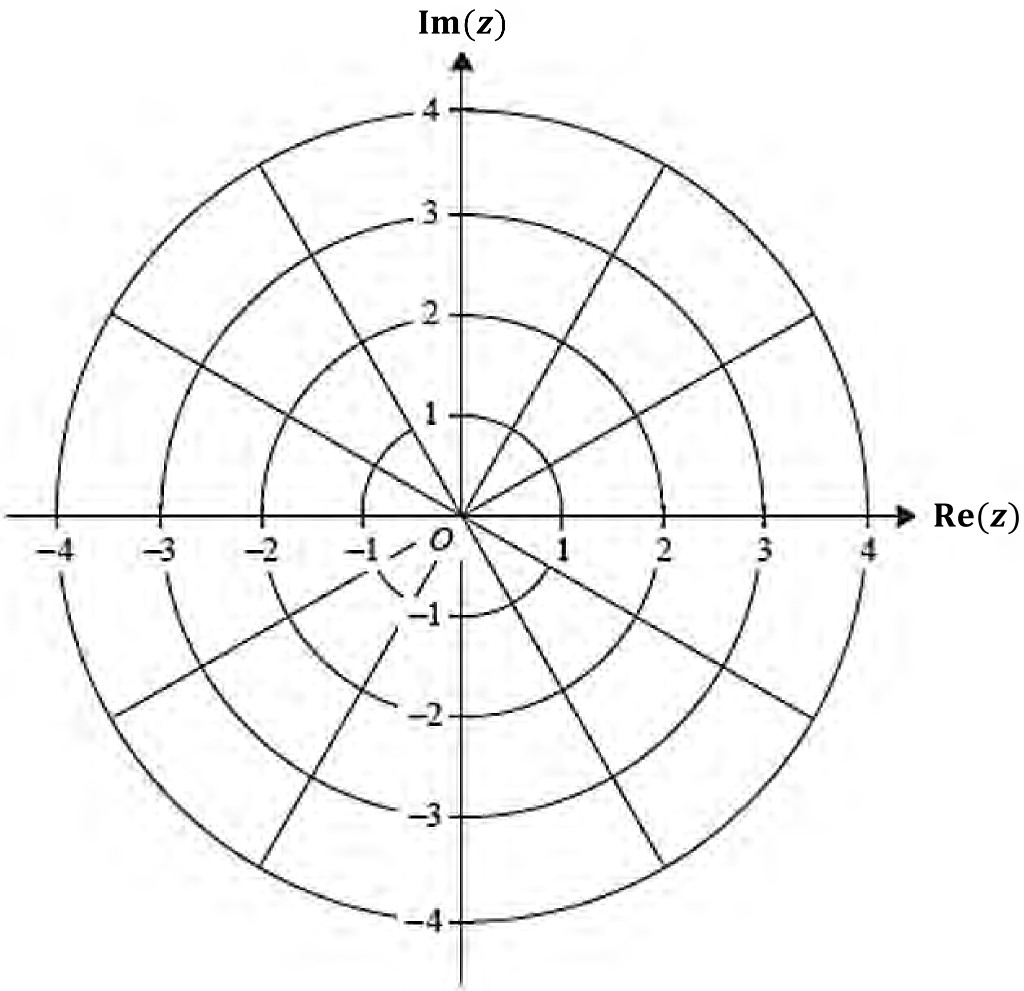
(iii) Given that $I_k > I_{k+1}$, deduce that

$$\frac{\pi}{2}\left(\frac{2n+1}{2n+2}\right) < \frac{2^2 \times 4^2 \times \cdots \times (2n)^2}{1 \times 3^2 \times 5^2 \times \cdots \times (2n-1)^2(2n+1)} < \frac{\pi}{2}$$ 2

END OF EXAMINATION

Answer Sheet for Q13 d) (i) and (ii)

Student Number:
Teacher:



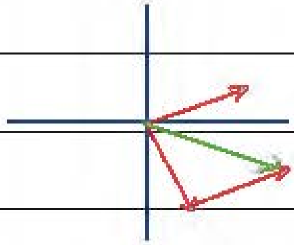
PLACE THIS ANSWER SHEET INSIDE YOUR BOOKLET FOR Q13

MARKER'S COMMENTS

EXT2 TRIAL SOLUTIONS

MULTIPLE CHOICE

1. $3e^{-i\pi/3} + 3e^{i\pi/6}$ (D)



2. (B)

3. $\arg(z\bar{z}) - \arg(z-\bar{z})$

$= \arg(x^2+y^2) - \arg(2yi)$ (A)

$= 0 - \frac{\pi}{2}$

4. $\int \frac{dx}{\sqrt{8-(x^2+2x+1)+1}} = \int \frac{dx}{9-(x+1)^2}$ (D)

$= \sin^{-1}\frac{x+1}{3} + c$

5. (B)

6. (B) odd f^n $\times$ even f^n = odd

$\therefore$ zero

7. (B) $(-2\ln\alpha + 8i)\,\text{cis}\,\frac{\pi}{2}$

$(-2\ln\alpha + 8i)\,i$

$= -8 - 2\ln\alpha\, i$

$C(-8, -2\ln\alpha)$ $\quad$ $A\left((\ln\alpha)^3, (\ln\alpha)^2\right)$

$-2\ln\alpha = (\ln\alpha)^2$

$(\ln\alpha)^3 = -8$ $\quad$ $(\ln\alpha)^2 + 2\ln\alpha = 0$

$\ln\alpha = -2$ $\quad$ $\ln\alpha(\ln\alpha + 2) = 0$

$e^{\ln\alpha} = e^{-2}$ $\quad$ $\ln\alpha = 0$ $\quad$ $\ln\alpha = -2$

$\alpha = e^{-2}$ $\quad$ $\alpha = e^0$ $\quad$ $\alpha = e^{-2}$

$\alpha = 1$

$\therefore \alpha = e^{-2}$

MARKER'S COMMENTS

8. (D) $\int_0^{\ln 2} \frac{1}{1+e^x}\,dx$

$u = 1+e^x$

$\frac{du}{dx} = e^x$

$\frac{dx}{du} = e^{-x} = \frac{1}{e^x}$

$dx = \frac{1}{u-1}\,du$

$x = \ln 2,\ u = 1+e^{\ln 2} = 3$

$x = 0,\ u = 1+e^0 = 2$

$= \int_2^3 \frac{1}{u} \times \frac{1}{u-1}\,du$

$= \int_2^3 \frac{1}{u(u-1)}\,du$

$= \int_2^3 \left(\frac{1}{u-1} - \frac{1}{u}\right) du$

$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$

$1 = A(u-1) + Bu$

$u = 1 \qquad u = 0$

$1 = B \qquad 1 = -A$

$A = -1$

$\frac{1}{u(u-1)} = \frac{-1}{u} + \frac{1}{(u-1)}$

9. $B(1,1,-1)$ and $A(1,-1,1)$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$

$\cos\theta = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$

$= \frac{\begin{pmatrix}1\\1\\-1\end{pmatrix} \cdot \begin{pmatrix}1\\-1\\1\end{pmatrix}}{\sqrt{3} \times \sqrt{3}}$

$= \frac{1-1-1}{3}$

$\cos\theta = -\frac{1}{3}$

$\theta = \pi - \cos^{-1}\left(\frac{1}{3}\right) \quad \therefore$ (C)

10. (A)

MARKER'S COMMENTS Question 11

Question 11

a) $z = 3-2i \qquad \omega = -1-i$

i) $z\bar{\omega} = (3-2i)(-1+i)$

$= -3+3i+2i+2$

$= -1+5i$

1 mark – for the conjugate

1 mark – for the answer (2)

Mostly this was well done

ii) arg ω is in quadrant 3

$= -(\pi - \tan^{-1}(1))$

$= -(\pi - \frac{\pi}{4})$

$= -\frac{3\pi}{4}$

1 mark for correct answer (1)

Some students did not have the answer in the correct quadrant or it was calculated incorrectly

iii) $\omega^{-12} = \left[\sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)\right]^{-12}$

$= 2^{-6}\,\text{cis}\,9\pi$

$= \frac{1}{64}\,\text{cis}\,\pi$

$= \frac{1}{64}(\cos\pi + i\sin\pi)$

$= -\frac{1}{64}$

1 mark – using De Moivre's theorem

1 mark – correct answer (2)

Mostly well done

MARKER'S COMMENTS

Question 11

b) i) $\vec{AB} = \begin{pmatrix} 3 \\ -3 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ -6 \\ -3 \end{pmatrix}$

$|\vec{AB}| = \sqrt{2^2 + (-6)^2 + (-3)^2}$

$= \sqrt{49} = 7$

1 mark for $\vec{AB}$, 1 mark for $|\vec{AB}|$

Well done

2

ii) $\frac{1}{7}\begin{pmatrix} 2 \\ -6 \\ -3 \end{pmatrix} \times 14$

$= 2\begin{pmatrix} 2 \\ -6 \\ -3 \end{pmatrix}$

$= 4\underset{\sim}{i} - 12\underset{\sim}{j} - 6\underset{\sim}{k}$

1 mark - Various forms of correct answers given

1

iii) $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -6 \\ -3 \end{pmatrix}$

1 mark for the correct point

1 mark for the correct equation

Mostly well done

2

MARKER'S COMMENTS Question 11

c) $z^2 - z + (4-2i) = 0$

$$z = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(4-2i)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{1 - 16 + 8i}}{2} \quad \text{------} \quad (1)$$

$$= \frac{1 \pm \sqrt{-15 + 8i}}{2} \quad \text{-----} \quad (2)$$

Method 1 : From (2)

Consider

$$a + ib = \sqrt{-15+8i}$$

$$(a+ib)^2 = -15 + 8i$$

$$a^2 + 2abi - b^2 = -15 + 8i$$

$$a^2 - b^2 + 2abi = -15 + 8i$$

Equating parts

$$a^2 - b^2 = -15 \quad \text{---} \quad (3) \qquad 2ab = 8$$

$$b = \frac{4}{a} \quad \text{---} \quad (4)$$

Sub in (3)

$$a^2 - \left(\frac{4}{a}\right)^2 = -15$$

$$a^2 - \frac{16}{a^2} + 15 = 0$$

$$a^4 + 15a^2 - 16 = 0$$

$$(a^2 + 16)(a^2 - 1) = 0$$

but $a^2 = -16$, $a \in \mathbb{R}$ $\therefore$ $a^2 = 1$

$$a = \pm 1$$

MARKER'S COMMENTS Question 11

$\therefore a=1, \quad b=4 \quad$ and $\quad a=-1, \quad b=-1$

$\therefore 1+4i \quad , \quad -1-4i$

$\therefore \pm(1+4i)$

$\therefore z = \dfrac{1 \pm (1+4i)}{2}$

$z = \dfrac{1+1+4i}{2} \quad$ or $\quad z = \dfrac{1-1-4i}{2}$

$= \dfrac{2+4i}{2} \qquad\qquad = -2i$

$z = 1+2i$

$\therefore z = 1+2i \, , \, -2i$

3 marks for the correct answer

2 marks for showing $z = \dfrac{1 \pm (1+4i)}{2}$

Mostly well done. Various methods were used

Method 2: From ①

$z = \dfrac{1 \pm \sqrt{1-16+8i}}{2}$

$= \dfrac{1 \pm \sqrt{1+(4i)^2+8i}}{2}$

$= \dfrac{1 \pm \sqrt{(1+4i)^2}}{2}$

$= \dfrac{1 \pm (1+4i)}{2}$

then as per above

MARKER'S COMMENTS

Question 11

d) $\int_0^{\pi} \sin^3 x \cos^2 x \, dx$

$= \int_0^{\pi} \sin x \sin^2 x \cos^2 x \, dx$

Let $u = \cos x$

$\frac{du}{dx} = -\sin x$

$= -\int_0^{\pi} -\sin^2 x \cos^2 x \sin x \, dx$ — 1

when $x = 0$, $u = \cos 0 = 1$

$x = \pi$, $u = \cos \pi = -1$

$= -\int_0^{\pi} (1 - \cos^2 x) \cos^2 x (-\sin x) dx$

$= -\int_1^{-1} (1 - u^2) u^2 \, du$

$= \int_{-1}^{1} (u^2 - u^4) \, du$

$= \left[\frac{u^3}{3} - \frac{u^5}{5} \right]_{-1}^{1}$ — 1

$= \left(\frac{1}{3} - \frac{1}{5} \right) - \left(-\frac{1}{3} + \frac{1}{5} \right)$

$= \frac{1}{3} - \frac{1}{5} + \frac{1}{3} - \frac{1}{5}$

$= \frac{4}{15}$ — 1

Mostly well done, however some students did not seem familiar with the style of question

Made with Goodnotes

MARKER'S COMMENTS Question 11

Alternative method

$$\int_0^{\pi} \sin^3 x \cos^2 x \, dx$$

$$= \int_0^{\pi} \sin x . \sin^2 x \ \cos^2 x \, dx$$

$$= \int_0^{\pi} \sin x \left(1 - \cos^2 x\right) \cos x^2 \, dx \quad \text{——} \quad 1$$

$$= \int_0^{\pi} \sin x . \cos^2 x - \sin x \cos^4 x \, dx$$

$$= \left[-\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} \right]_0^{\pi} \quad \text{——} \quad 1$$

$$= \left[\left(-\frac{\cos^3 \pi}{3} + \frac{\cos^5 \pi}{5} \right) - \left(-\frac{\cos^3 0}{3} + \frac{\cos^5 0}{5} \right) \right]$$

$$= \left[\left(\frac{1}{3} - \frac{1}{5} \right) - \left(-\frac{1}{3} + \frac{1}{5} \right) \right]$$

$$= \frac{2}{15} - \left(\frac{-2}{15} \right)$$

$$= \frac{4}{15} \quad \text{——} \quad 1$$

QUESTION 12

MARKER'S COMMENTS

a) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -1 \\ 5 \end{pmatrix}$

$2 = 6 + 4\lambda$	$y = 2 - \lambda$	$z = 3 + 5\lambda$
$4\lambda = -4$	$y = 2 - (-1)$	$z = 3 - 5$
$\lambda = -1$	$y = 3$	$z = -2$

1 mark correct λ

$\frac{1}{2}$ mark correct x

$\frac{1}{2}$ mark correct y

/2

NOTES: Very well done

b) METHOD 1

$$\int \frac{x^3}{x^2+x+1}\,dx$$

$$= \int (x-1)\,dx + \int \frac{1}{x^2+x+1}\,dx$$

$$\begin{array}{r} x-1 \\ x^2+x+1 \enclose{longdiv}{x^3} \\ -\ \underline{x^3+x^2+x} \\ -x^2-x \\ -\ \underline{-x^2-x-1} \\ 1 \end{array}$$

$x^3 = (x^2+x+1)(x-1)+1$

$\frac{x^3}{x^2+x+1} = x-1 + \frac{1}{x^2+x+1}$

1 to split integral

$$= \int (x-1)\,dx \quad \int \frac{1}{x^2+x+\left(\frac{1}{2}\right)^2+1-\left(\frac{1}{2}\right)^2}\,dx$$

$$= \int (x-1)\,dx \quad \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$$

$$= \int (x-1)\,dx \quad \int \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2+\left(x+\frac{1}{2}\right)^2}\,dx$$

$\int \frac{f'(x)}{a^2+[f(x)]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a}+c$

1 setting up integral to enable you to use inv tan

$$= \frac{x^2}{2} - x + \frac{1}{\sqrt{3}/2}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right)+c$$

1 mark correct answer.

$$= \frac{x^2}{2} - x + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+c$$

/3

Q12 b) cont ....

MARKER'S COMMENTS

METHOD 2

$$\int \frac{x^3}{x^2+x+1}\,dx = \int \frac{x^3-1+1}{x^2+x+1}\,dx$$

① to split integral

$$= \int \frac{x^3-1}{x^2+x+1}\,dx + \int \frac{1}{x^2+x+1}\,dx$$

$$= \int \frac{(x-1)(x^2+x+1)}{x^2+x+1}\,dx + \int \frac{1}{x^2+x+\left(\frac{1}{2}\right)^2+1-\left(\frac{1}{2}\right)^2}\,dx$$

$$= \int (x-1)\,dx + \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

① setting up integral to enable you to use inv tan

$$= \frac{x^2}{2} - x + \frac{1}{\sqrt{3}/2}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right) + C$$

$$= \frac{x^2}{2} - x + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

① mark correct answer.

/3

MARKER'S COMMENTS

Q12 cont...

c) (i) Since the coefficients of each term are real, and there are at most three zeros and z_2, z_3 are complex numbers, then they are also conjugates

$\therefore$ z_2 and z_3 are complex conjugates.

① mark correct explanation

/1

(ii) METHOD 1

$P(z) = (z-z_1)(z-z_2)(z-z_3)$

$P(2) = {}^-13$ $\quad |z_2+z_3| = 0$ $\quad |z_2-z_3| = 6$

$|a+ib+a-ib| = 0$ $\quad |a+ib-(a-ib)| = 6$

$|2a| = 0$ $\quad |a+ib-a+ib| = 6$

$\therefore a = 0$ $\quad |2bi| = 6$

$\pm 2b = 6$

$b = \pm 3$

$\therefore$ $z_2 = 3i$, $z_3 = -3i$

① mark to find z_2 and z_3.

Since $P(2) = {}^-13$

${}^-13 = (2-z_1)(2-3i)(2+3i)$

${}^-13 = (2-z_1)(4+9)$

$-1 = 2-z_1$

① mark to find z_1

$z_1 = 3$

$\therefore$ $P(z) = (z-3)(z-3i)(z+3i)$

$P(z) = z^3+az^2+bz+c$ and $P(z) = (z-3)(z^2+9)$

$= z^3+9z-3z^2-27$

$= z^3-3z^2+9z-27$

① mark correct a, b, c

/3

$\therefore$ $a=-3$, $b=9$, $c=-27$

Q12 cont ...

MARKER'S COMMENTS

METHOD 2

$P(2) = -13$

$-13 = 2^3 + a(2)^2 + 2b + c$

$-13 = 8 + 4a + 2b + c$

$4a + 2b + c = -21$

Sum of roots one at a time

$\alpha + \beta + \gamma = -\frac{b}{a}$

$z_1 + z_2 + z_3 = -\frac{a}{1}$

$z_1 + 3i - 3i = -a$

$z_1 = -a$

$a = -z_1$

$z_1 = -a$

product

$\alpha\beta\gamma = -\frac{d}{a}$

$z_1(3i)(-3i) = -c$

$9z_1 = -c$

$z_1 = \frac{-c}{9}$

$-a = -\frac{c}{9}$

$a = \frac{c}{9}$

Sum of roots two at a time

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$

$z_1(3i) + z_1(-3i) + (3i)(-3i) = b$

$z_1(3i - 3i) + 9 = b$

$b = 9$

$4a + 2b + c = -21$

$4\left(\frac{c}{9}\right) + 2(9) + c = -21$

$4c + 162 + 9c = -189$

$13c = -351$

$c = -27$

$\therefore a = \frac{-27}{9}$

$a = -3$

$a = -3, \quad b = 9, \quad c = -27$

Q12 cont ...

MARKER'S COMMENTS

d)

$|2x+1| \le 5 \Rightarrow |x| \le 2$

Counterexample $x = -3$

Does $|2x-3+1| \le 5 \Rightarrow |-3| \le 2$?

Does $5 \le 5 \Rightarrow 3 \le 2$?

$\therefore$ No, this is false

$\therefore$ $|2x+1| \le 5$ does not imply $|x| \le 2$.

Q12 cont ...

MARKER'S COMMENTS

e) (i)

$$\underset{\sim}{r}_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \qquad \underset{\sim}{r}_2 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$$

CHECK:

$2 + \lambda = 2 + \mu$ $\qquad$ $1 + 2\lambda = -1 + 4\mu$ $\qquad$ $1 + \lambda = -1 + 3\mu$

$\lambda = \mu$ $\qquad$ $1 + 2\lambda = -1 + 4\lambda$ $\qquad$ $1 + \lambda = -1 + 3\lambda$

$2\lambda = 2$ $\qquad$ $1 + 1 = -1 + 3$

$\lambda = 1$ $\qquad$ $2 = 2$ ✓

$\therefore \lambda = 1, \mu = 1$

$\lambda = 1$

$$\therefore r_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix}$$

$\mu = 1$

$$r_2 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix}$$

1 Find λ and μ

$\therefore \underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ meet at $\begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix}$ $\therefore$ Not skew

1 Find point of intersection and justifying the lines are not skew

/2

Q12 e) cont...

MARKER'S COMMENTS

(ii) Direction vector of $\underset{\sim}{r}_1$ is $\begin{pmatrix}1\\2\\1\end{pmatrix}$ and the second line is $\begin{pmatrix}1\\4\\3\end{pmatrix}$

Let $\begin{pmatrix}a\\b\\c\end{pmatrix}$ be the direction vector of the new line

$\therefore \begin{pmatrix}a\\b\\c\end{pmatrix}\cdot\begin{pmatrix}1\\2\\1\end{pmatrix}=0 \qquad \begin{pmatrix}a\\b\\c\end{pmatrix}\cdot\begin{pmatrix}1\\4\\3\end{pmatrix}=0$

1 mark set set up dot product with direction vectors

$a+2b+c=0 \qquad a+4b+3c=0$

$$\begin{array}{r} a+2b+c=0 \\ a+4b+3c=0 \\ \hline -2b-2c=0 \\ b=-c \end{array} \quad - \quad \begin{array}{r} 2a+4b+2c=0 \\ a+4b+3c=0 \\ \hline a-c=0 \\ a=c \end{array} \; -$$

$\therefore \begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}c\\-c\\c\end{pmatrix}=c\begin{pmatrix}1\\-1\\1\end{pmatrix}$

1 mark find new direction vector.

$\therefore$ direction vector of new line is $\begin{pmatrix}1\\-1\\1\end{pmatrix}$

and point is $\begin{pmatrix}3\\3\\2\end{pmatrix}$

$\therefore \underset{\sim}{r}_3=\begin{pmatrix}3\\3\\2\end{pmatrix}+\lambda\begin{pmatrix}-1\\-1\\1\end{pmatrix}$

1 mark for correct equation of $\perp$ line.

/3

MARKER'S COMMENTS Question 13

a) i) $\dfrac{3x^2+4x+12}{x(x^2+4)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+4}$

$\therefore \quad 3x^2+4x+12 = A(x^2+4) + x(Bx+C)$

when $x=0$

$12 = 4A$

$\therefore \quad A = 3$

when $x=1$

$3+4+12 = 5A + B + C$

$19 = 15 + B + C \quad (\because \; A=3)$

$\therefore \; B+C = 4 \quad \text{---} \; 1$

when $x=-1$

$3-4+12 = 5A + B - C$

$11 = 15 + B - C$

$B - C = -4 \quad \text{---} \; 2$

$1 + 2, \quad 2B = 0$

$B = 0 \quad \text{sub in } 2$

$-C = -4, \; C = 4$

$\therefore \quad \dfrac{3x^2+4x+12}{x(x^2+4)} = \dfrac{3}{x} + \dfrac{4}{x^2+4}$

2 marks – correct answer

1 mark – for finding $A=3$

or – for B and C correct

Mostly well done. Various methods used

Made with Goodnotes

MARKER'S COMMENTS **Question 13**

a)i) Alternative Method

$$3x^2 + 4x + 12 \equiv A(x^2+4) + x(Bx+C)$$
$$= Ax^2 + 4A + Bx^2 + Cx$$
$$= x^2(A+B) + Cx + 4A$$

Equating parts

$4A = 12$

$A = 3$ ———— 1 mark

$A + B = 3$

$3 + B = 3$

$B = 0$, $C = 4$ ———— 1 mark

2

ii) $$\int \frac{3x^2+4x+12}{x(x^2+4)}\,dx = \int\left(\frac{3}{x} + \frac{4}{x^2+4}\right)dx$$
$$= 3\int \frac{1}{x}\,dx + 4\int \frac{1}{x^2+2^2}\,dx$$
$$= 3\ln|x| + 4 \times \frac{1}{2}\tan^{-1}\frac{x}{2} + c$$
$$= 3\ln|x| + 2\tan^{-1}\frac{x}{2} + c$$

1 mark – Integrating $\int \frac{3}{x}\,dx$ correctly as $3\ln|x|$

1 mark – " $\int \frac{4}{x^2+4}\,dx$ correctly

2

½ mark for not having the absolute value when integrating $\int \frac{3}{x}\,dx$.

Made with Goodnotes

MARKER'S COMMENTS Question 13

b) i) Contrapositive of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$

$\therefore$ 'If x is not odd then x^2+3x+1 is not even' — 1mark

Well done by most

1

ii) Let $x = 2m$ where $m \in \mathbb{Z}$

$$x^2+3x+1 = (2m)^2+3(2m)+1$$
$$= 4m^2+6m+1$$
$$= 2(2m^2+3m)+1$$

which is odd

$\therefore$ If x is even then x^2+3x+1 is odd

$\therefore$ By contrapositive if x^2+3x+1 is even, then x is odd.

1 mark – using the substitution $x=2m$

1 mark – successfully proving it with strong appropriate conclusion.

2

Well done by most.

MARKER'S COMMENTS Question 13

c) $I_n = \int e^{-x}\sin x\,dx$ Let $u=\sin x$ $v'=e^{-x}$

$u'=\cos x$ $v=-e^{-x}$

$$= -e^{-x}\sin x - \int -e^{-x} \times \cos x\,dx$$
$$= -e^{-x}\sin x + \int e^{-x}\cos x\,dx$$ —— 1mark

Let $u=\cos x$ $v'=e^{-x}$

$u'=-\sin x$ $v=-e^{-x}$

$$I_n = -e^{-x}\sin x + \left[-e^{-x}\cos x - \int -e^{-x} \times -\sin x\,dx\right]$$ —— 1mark

$$= -e^{-x}\sin x - e^{-x}\cos x - \int e^{-x}\sin x\,dx$$
$$= -e^{-x}\sin x - e^{-x}\cos x - I_n$$
$$2I_n = -e^{-x}\sin x - e^{-x}\cos x$$
$$= -e^{-x}(\sin x + \cos x) + c$$
$$I_n = -\frac{e^{-x}}{2}(\sin x + \cos x) + c$$ —— 1 mark

3

1 mark – applying once

1 mark – applying again

1 mark – correct answer

Some students had problems with the negative (–) sign.

2 marks were given if students wrote

$$-\frac{e^{-x}}{2}(\sin x - \cos x)$$

MARKER'S COMMENTS Question 13

d) i) $\arg(z - z_4) = \frac{5\pi}{6}$

$z_4 = \sqrt{3} + i$

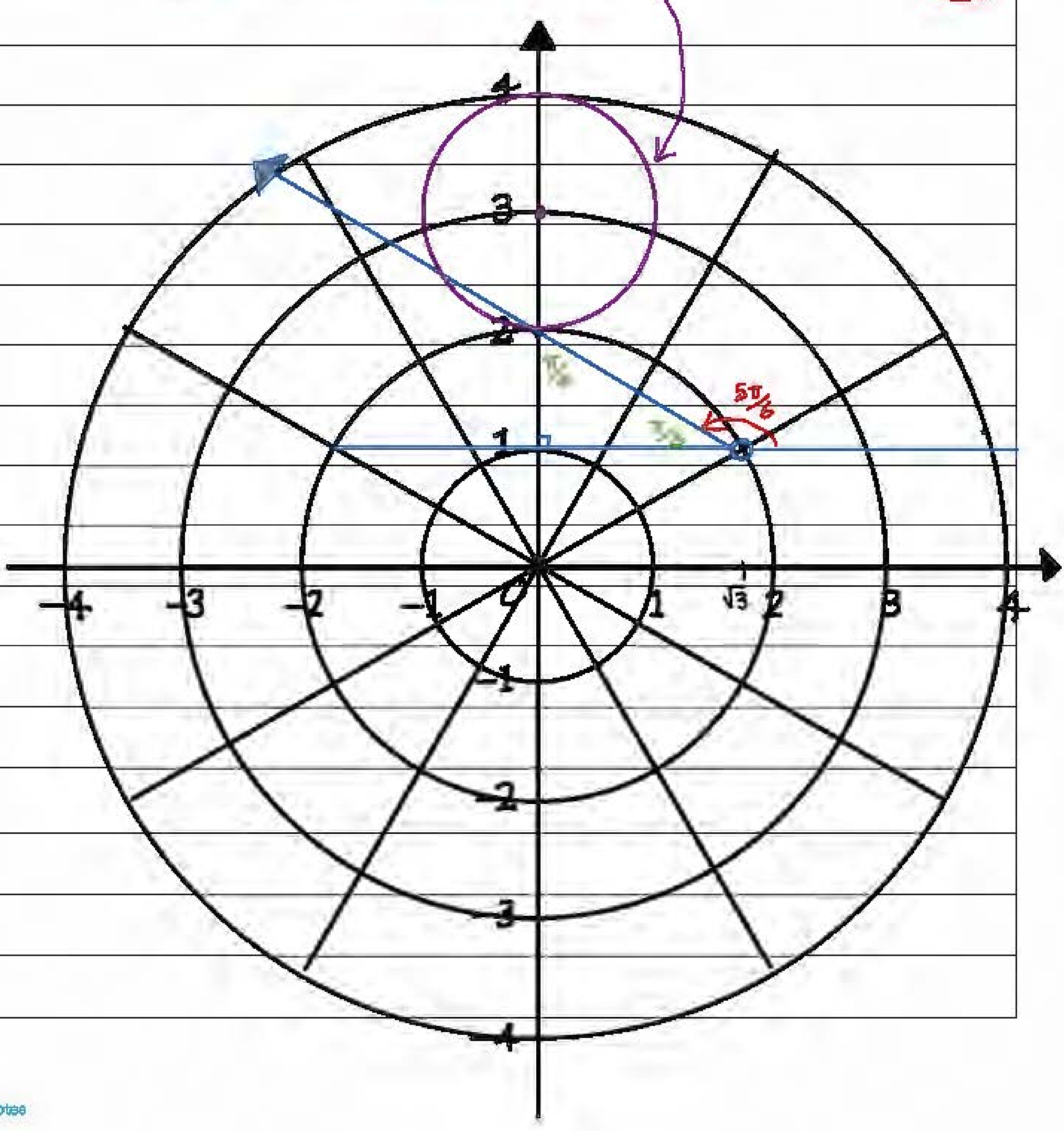
$\therefore \arg[z - (\sqrt{3}+i)] = \frac{5\pi}{6}$

1 mark – correct open circle on $(\sqrt{3}, 1)$

1 mark – correct ray (2)

ii) $|z - 3i| = 1$

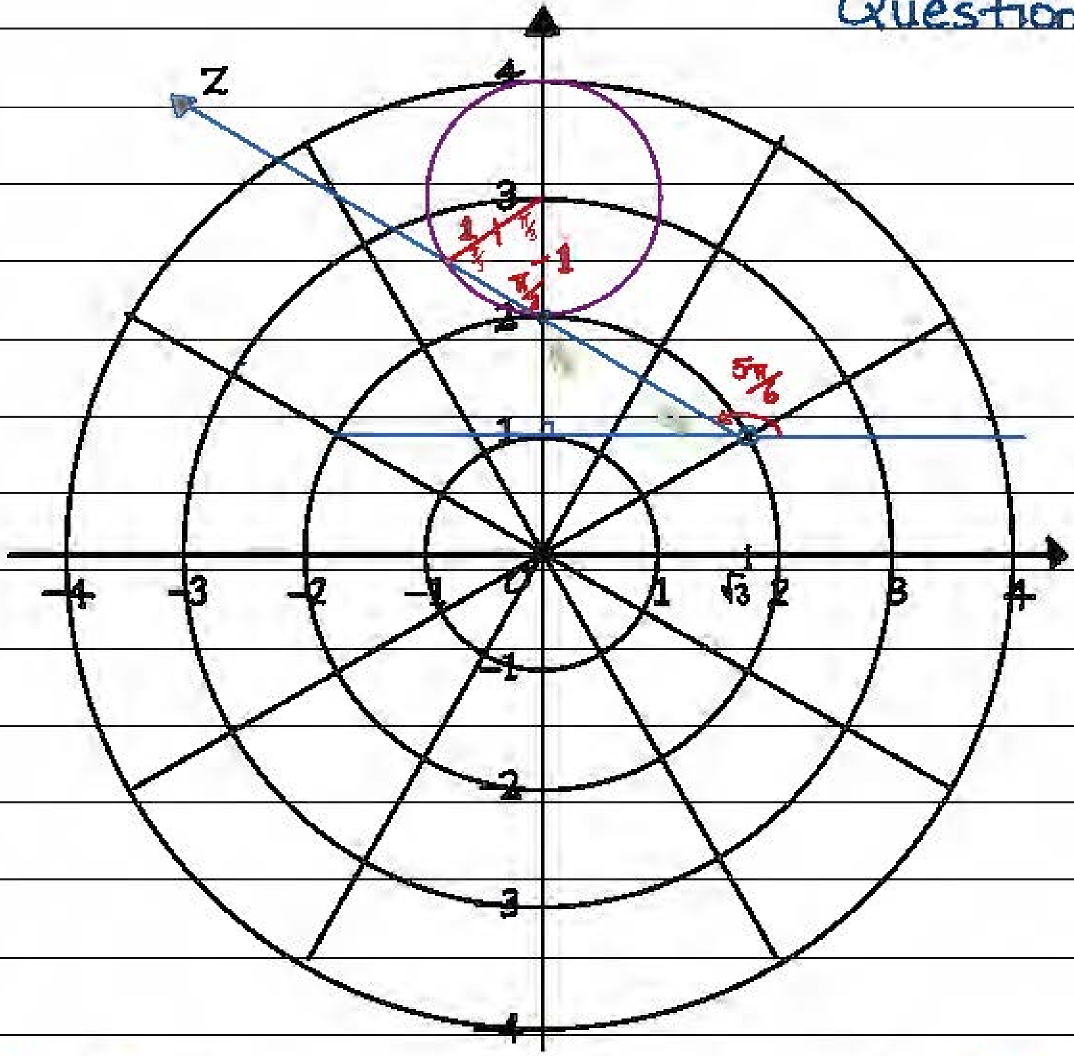
1 mark – correct circle (1)



MARKER'S COMMENTS

Question 13

iii)



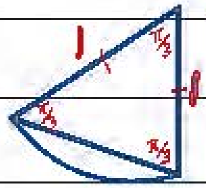
Equation of $z - z_4$: $m = \tan\theta$ and $(\sqrt{3}, 1)$

$= \tan 5\pi/6$

$= -\tan \pi/6 = -\frac{1}{\sqrt{3}}$

$\therefore\ y - 1 = -\frac{1}{\sqrt{3}}(x - \sqrt{3})$ or $y = -\frac{1}{\sqrt{3}}x + 2$ — ½ mk

when $x = 0$, $y = 2$ — ½ mark



Area of sector $= \frac{1}{2}r^2\theta$

$= \frac{1}{2} \times 1^2 \times \pi/3$

$= \pi/6$

Area of $\Delta = \frac{1}{2}ab\sin C$

$= \frac{1}{2} \times 1 \times 1 \sin\frac{\pi}{3}$

$= \frac{1}{2} \times \frac{\sqrt{3}}{2}$

$\therefore$ Area of minor segment $= \frac{\pi}{6} - \frac{\sqrt{3}}{4}$

$= \frac{2\pi - 3\sqrt{3}}{12}$ — 1 mark

2

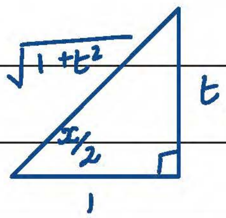
Those who used simple geometric methods for angles and sides in triangles had more success.

QUESTION 14

MARKER'S COMMENTS

a) METHOD 1

Let $t = \tan\frac{x}{2}$



$\tan\frac{x}{2} = \frac{t}{1}$

$$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$$

$$= \frac{1}{2} \times \frac{\sqrt{1+t^2}}{1}$$

$$\frac{dt}{dx} = \frac{\sqrt{1+t^2}}{2}$$

$$\frac{dx}{dt} = \frac{2}{\sqrt{1+t^2}}$$

① mark correctly establishes result.

METHOD 2

Let $t = \tan\frac{x}{2}$

$$\frac{x}{2} = \tan^{-1}t$$

$$x = 2\tan^{-1}t$$

$$\frac{dx}{dt} = \frac{2}{1+t^2}$$

/1

MARKER'S COMMENTS

Q14 b) cont...

(ii)

$\therefore \sin x = \frac{2t}{1+t^2} \qquad \cos x = \frac{1-t^2}{1+t^2}$

when $x = \frac{\pi}{2} \qquad x = 0$

$\sin\frac{\pi}{2} = \frac{2t}{1+t^2} \qquad \sin 0 = \frac{2t}{1+t^2}$

$1 = \frac{2t}{1+t^2} \qquad 2t = 0$

$t = 0$

1 boundries and set up of integral

$1+t^2 = 2t$

$t^2 - 2t + 1 = 0$

$(t-1)^2 = 0$

$t = 1$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x + \cos x}\, dx$$

$$= \int_0^1 \frac{1}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2}\, dt$$

$$= \int_0^1 \frac{1+t^2}{1+t^2+2t+1-t^2} \times \frac{2}{1+t^2}\, dt$$

$$= \int_0^1 \frac{\cancel{2}}{\cancel{2}(t+1)}\, dt$$

1 mark evaluating integral correctly

$$= \left[\ln|t+1|\right]_0^1$$

$$= \ln 2 - \ln 1$$

/2

$$= \ln 2$$

MARKER'S COMMENTS

$(x-y)^2 > 0$ as $x \neq y$.

$x^2 - 2xy + y^2 > 0$

$x^2 + y^2 > 2xy$

$x \to x^2$, $y \to y^2$ $\quad$ $x \to y^2$, $y \to z^2$ $\quad$ $x \to x^2$, $y \to z^2$

$x^4 + y^4 > 2x^2y^2 \ldots ①$ $\quad$ $x^4 + z^4 \geq 2x^2z^2 \ldots ②$ $\quad$ $y^4 + z^4 > 2y^2z^2 \ldots ③$

1 mark using previously known inequality

Sum $① + ② + ③$

$x^4 + y^4 + y^4 + z^4 + x^4 + z^4 > 2x^2y^2 + 2y^2z^2 + 2x^2z^2$

$2x^4 + 2y^4 + 2z^4 > 2(x^2y^2 + y^2z^2 + x^2z^2)$

$2(x^4 + y^4 + z^4) > 2(x^2y^2 + y^2z^2 + x^2z^2)$

$\therefore x^4 + y^4 + z^4 > x^2y^2 + x^2z^2 + z^2y^2$

1 mark correcty proving result.

MARKER'S COMMENTS

Q14 cont…

c)

S_2 $x^2+y^2+z^2-12x-12y-16z+100=0$

$x^2-12x+\left(\frac{-12}{2}\right)^2+y^2-12y+\left(\frac{-12}{2}\right)^2+z^2-16z+\left(\frac{-16}{2}\right)^2=-100+\left(\frac{-12}{2}\right)^2+\left(\frac{-12}{2}\right)^2+\left(\frac{-16}{2}\right)^2$

$(x-6)^2+(y-6)^2+(z-8)^2=-100+36+36+64$

$(x-6)^2+(y-6)^2+(z-8)^2=36$

1 mark establishin centre and radius for S_2

$\therefore c(6,6,8)$ $r=6$

S_1 $c(2,2,1)$ passes through $O(0,0,0)$

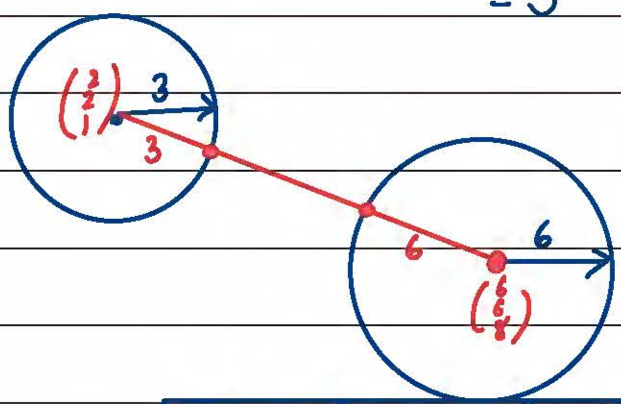
$\therefore$ radis $\sqrt{(2-0)^2+(2-0)^2+(1-0)^2}$

$=\sqrt{4+4+1}$

$=\sqrt{9}$

$=3$

½ establishing radius for S_1



If the distance between the two radii is less than or equal to 9 then they intersect. If it is exactly 9, they touch.

1 providing reason / justification for finding distance between the two centres and the link to why they touch or intersect.

$\sqrt{(6-2)^2+(6-2)^2+(8-1)^2}$

$=\sqrt{16+16+49}$

$=\sqrt{81}$

$=9$

1 mark finding distance between centres

Since the distance between them is $3+6=9$ then they touch.

½ mark stating they touch.

MARKER'S COMMENTS

d) <u>Method 1</u>

Assume $\exists\, a \geq 2$ such that $\sqrt{a} + \sqrt{a+2} \leq \sqrt{a+8}$

$\therefore \left(\sqrt{a} + \sqrt{a+2}\right)^2 \leq \left(\sqrt{a+8}\right)^2$

$a + 2\sqrt{a(a+2)} + a + 2 \leq a + 8$

$2a + 2\sqrt{a^2+2a} + 2 \leq a + 8$

$a + 2\sqrt{a^2+2a} - 6 \leq 0$

$\left(2\sqrt{a^2+2a}\right)^2 \leq (6-a)^2$

$4(a^2+2a) \leq 36 - 12a + a^2$

$4a^2 + 8a \leq 36 - 12a + a^2$

$3a^2 + 20a - 36 \leq 0$

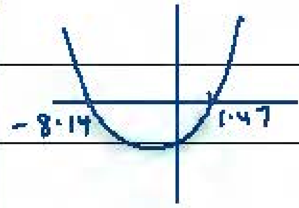
(this would the mean $\forall\, a \geq 2$, $\sqrt{a} + \sqrt{a+2} > \sqrt{a+8}$ would be false but if we can prove there are no values $a \geq 2$ such that $\sqrt{a} + \sqrt{a+2} \leq \sqrt{a+8}$ then the original statement must be true.

Solve $3a^2 + 20a - 36 = 0$

$a = \dfrac{-20 \pm \sqrt{20^2 - 4(3)(-36)}}{2(3)}$

$a = \dfrac{-20 \pm \sqrt{832}}{6}$

$a \doteqdot 1.47,\ -8.14$



$\therefore -8.14 < a < 1.47$

$\therefore a \ngeq 2$

So, the negation is false

$\therefore$ By contradiction $\sqrt{a} + \sqrt{a+2} > \sqrt{a+8}$ for all $a \geq 2$.

1 mark for assumption

1 mark to establish quadratic inequality

1 mark for solving it with appropriate conclusion.

/3

MARKER'S COMMENTS

Q14 continued.

Method 2

d) Assume $\exists\, a \geq 2$ such that $\sqrt{a} + \sqrt{a+2} \leq \sqrt{a+8}$

(this would the mean $\forall\, a \geq 2$, $\sqrt{a} + \sqrt{a+2} > \sqrt{a+8}$ would be fake BUT if we can prove there are NO values $a \geq 2$ such that $\sqrt{a} + \sqrt{a+2} \leq \sqrt{a+8}$ then the original statement must be true.

Consider:

$LHS^2 - RHS^2$

$= (\sqrt{a} + \sqrt{a+2})^2 - (\sqrt{a+8})^2$

$= a + 2\sqrt{a(a+2)} + a + 2 - a - 8$

$= a + 2\sqrt{a(a+2)} - 6$

The lowest value this could be for $a \geq 2$ is $a = 2$ as $a > 0$, $2\sqrt{a(a+2)} > 0$ and you are adding a to $2\sqrt{a(a+2)}$ and always subtracting the same value of 6.

when $a = 2$

$= 2 + 2\sqrt{2(2+2)} - 6$

$= 2 + 4\sqrt{2} - 6$

$= 4\sqrt{2} - 4$

$\doteqdot 1.66$ all other answers will be > than 1.66

$\therefore LHS^2 - RHS^2 \geq 1.66$

$\therefore LHS^2 - RHS^2 > 0$

$LHS^2 > RHS^2$

$\therefore LHS > RHS$

$\therefore \sqrt{a} + \sqrt{a+2} > \sqrt{a+8}$ for $a \geq 2$

$\therefore$ Contradiction for all $a \geq 2$

$\therefore$ There does not exist $a \geq 2$

$\therefore$ By contradiction $\sqrt{a} + \sqrt{a+2} > \sqrt{a+8}$ for all $a \geq 2$.

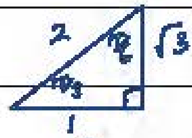
1 mark for assumption

1 mark considering $LHS^2 - RHS^2$ and justification for subbing in $a = 2$

1 mark to appropriatley and correctly finish proof.

Q14 cont...

MARKER'S COMMENTS



e)

$$\int_1^{\sqrt{3}} \frac{1}{x^2\sqrt{1+x^2}}\,dx \qquad \text{let } x = \tan\theta \qquad \frac{dx}{d\theta} = \sec^2\theta \checkmark$$

$x=\sqrt{3}$, $\tan\theta = \sqrt{3}$, $\theta = \frac{\pi}{3}$ ✓

$x = 1$, $\tan\theta = 1$, $\theta = \frac{\pi}{4}$ ✓

$dx = \sec^2\theta \, d\theta$

$$= \int_{\pi/4}^{\pi/3} \frac{1}{\tan^2\theta\sqrt{1+\tan^2\theta}} \times \sec^2\theta \, d\theta \checkmark$$

$\cos^2\theta + \sin^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$

$$= \int_{\pi/4}^{\pi/3} \frac{1}{\tan^2\theta\sqrt{\sec^2\theta}} \sec^2\theta \, d\theta$$

② marks to correctly set up integral

$$= \int_{\pi/4}^{\pi/3} \frac{\sec\theta}{\tan^2\theta}\, d\theta$$

① mark for $\int_{\pi/4}^{\pi/3} \frac{\cos\theta}{\sin^2\theta} d\theta$

$$= \int_{\pi/4}^{\pi/3} \frac{1}{\cos\theta} \times \frac{\cos^2\theta}{\sin^2\theta}$$

$$= \int_{\pi/4}^{\pi/3} \frac{\cos\theta}{\sin^2\theta}\, d\theta$$

let $u = \sin\theta$, $\frac{du}{d\theta} = \cos\theta$, $du = \cos\theta \, d\theta$

$\theta = \frac{\pi}{3}$, $u = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\theta = \frac{\pi}{4}$, $u = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$= \int_{1/\sqrt{2}}^{\sqrt{3}/2} u^{-2}\, du$$

$$= \left[\frac{u^{-1}}{-1}\right]_{1/\sqrt{2}}^{\sqrt{3}/2}$$

$$= \left[-\frac{1}{u}\right]_{1/\sqrt{2}}^{\sqrt{3}/2}$$

$$= -\left[\frac{2}{\sqrt{3}} - \frac{\sqrt{2}}{1}\right]$$

$$= -\left(\frac{2-\sqrt{6}}{\sqrt{3}}\right)$$

$$= \frac{\sqrt{6}-2}{\sqrt{3}}$$

OR

$$\int_{\pi/4}^{\pi/3} \operatorname{cosec}\theta\cot\theta \, d\theta$$

$$= \left[-\operatorname{cosec}\theta\right]_{\pi/4}^{\pi/3}$$

$$= -\left[\operatorname{cosec}\frac{\pi}{3} - \operatorname{cosec}\frac{\pi}{4}\right]$$

$$= -\left(\frac{2}{\sqrt{3}} - \sqrt{2}\right)$$

$$= \sqrt{2} - \frac{2}{\sqrt{3}}$$

$$= \frac{\sqrt{6}-2}{\sqrt{3}}$$

① mark to get correct answer

/4

MARKER'S COMMENTS Question 15

a) i) Consider RHS − LHS

$$\frac{2k+2}{k+3} - \frac{2k}{k+2}$$

$$= \frac{2(k+1)(k+2) - 2k(k+3)}{(k+3)(k+2)}$$

$$= \frac{2(k^2+3k+2) - 2k^2 - 6k}{(k+3)(k+2)}$$

$$= \frac{\cancel{2k^2} + \cancel{6k} + 4 - \cancel{2k^2} - 6k}{(k+3)(k+2)}$$

$$= \frac{4}{(k+3)(k+2)}$$

>0 as $k>0$ $\therefore$ $k+3>0$ and $k+2>0$

$$\therefore \frac{2k+2}{k+3} - \frac{2k}{k+2} > 0$$

2

$$\therefore \frac{2k}{k+2} < \frac{2k+2}{k+3}$$

Students who made one side of the inequality 0 had more success.

ii) Prove true for $n=1$

$$\text{LHS} = \frac{1}{3!} + \frac{2}{4!} + \frac{3}{5!} + \cdots + \frac{n}{(n+2)!} \qquad \text{RHS} = \frac{2n}{n+2} - \frac{1}{(n+2)!}$$

For $n=1$, $\text{LHS} = \frac{1}{3!} = \frac{1}{6}$

$$\text{RHS} = \frac{2(1)}{1+2} - \frac{1}{(1+2)!} = \frac{2}{3} - \frac{1}{3!} = \frac{2}{3} - \frac{1}{6} = \frac{1}{2}$$

1mk

Now $\frac{1}{6} < \frac{1}{2}$

i.e, LHS < RHS $\therefore$ true for $n=1$

Made with Goodnotes

MARKER'S COMMENTS Question 15

ii) Cont'd

Assume that the statement is true for $n=k, k\geq 1$

i.e $\frac{1}{3!}+\frac{2}{4!}+\frac{3}{5!}+\cdots+\frac{k}{(k+2)!}<\frac{2k}{k+2}-\frac{1}{(k+2)!}$ — (*)

Prove the statement is true for $n=k+1$

i.e prove $\frac{1}{3!}+\frac{2}{4!}+\frac{3}{5!}+\cdots+\frac{k}{(k+2)!}+\frac{k+1}{(k+3)!}<\frac{2(k+1)}{(k+3)}-\frac{1}{(k+3)!}$

$$LHS=\frac{1}{3!}+\frac{2}{4!}+\frac{3}{5!}+\cdots+\frac{k}{(k+2)!}+\frac{k+1}{(k+3)!}$$

$$<\frac{2k}{k+2}-\frac{1}{(k+2)!}+\frac{k+1}{(k+3)!}$$ by the assumption (*)

$$=\frac{2k}{k+2}-\frac{1}{(k+2)!}+\frac{k+2-1}{(k+3)!}$$

$$=\frac{2k}{k+2}-\frac{1}{(k+2)!}+\frac{k+2}{(k+3)!}-\frac{1}{(k+3)!}$$

$$=\frac{2k}{k+2}-\frac{k+3}{(k+2)!\,k+3}+\frac{k+2}{(k+3)!}-\frac{1}{(k+3)!}$$

$$=\frac{2k}{k+2}-\frac{k+3}{(k+3)!}+\frac{k+2}{(k+3)!}-\frac{1}{(k+3)!}$$

$$=\frac{2k}{k+2}-\left(\frac{k+3}{(k+3)!}-\frac{k+2}{(k+3)!}\right)-\frac{1}{(k+3)!}$$

$$=\frac{2k}{k+2}-\frac{k+3-k-2}{(k+3)!}-\frac{1}{(k+3)!}$$ — 1 mark

$$=\frac{2k}{k+2}-\frac{2}{(k+3)!}$$

$$<\frac{2k}{k+2}-\frac{1}{(k+3)!}$$ — 1 mark

$$<\frac{2(k+1)}{k+3}-\frac{1}{(k+3)!}$$ by (i) where $\frac{2k}{k+2}<\frac{2k+2}{k+3}$ 3

MARKER'S COMMENTS

Question 15

Alternative method for the inductive step

Prove the statement is true for $n=k+1$

i.e prove $\frac{1}{3!}+\frac{2}{4!}+\frac{3}{5!}+\cdots+\frac{k}{(k+2)!}+\frac{k+1}{(k+3)!}<\frac{2(k+1)}{(k+3)}-\frac{1}{(k+3)!}$

$LHS=\frac{1}{3!}+\frac{2}{4!}+\frac{3}{5!}+\cdots+\frac{k}{(k+2)!}+\frac{k+1}{(k+3)!}$

$<\frac{2k}{k+2}-\frac{1}{(k+2)!}+\frac{k+1}{(k+3)!}$ by the assumption (4)

$<\frac{2k+2}{k+3}-\frac{1}{(k+2)!}+\frac{k+1}{(k+3)!}$ from (i)

$=\frac{2k+2}{k+3}-\frac{[k+3-(k+1)]}{(k+3)!}$

$=\frac{2k+2}{k+3}-\frac{2}{(k+3)!}$

$<\frac{2(k+1)}{k+3}-\frac{1}{(k+3)!}$ as you are minusing less than what you did before

$\therefore$ true for $n=k+1$ when true for $n=k$.

By the principle of mathematical induction the statement is true for all integer $n\geq 1$.

Note: Most students had the case $n=1$ correct.

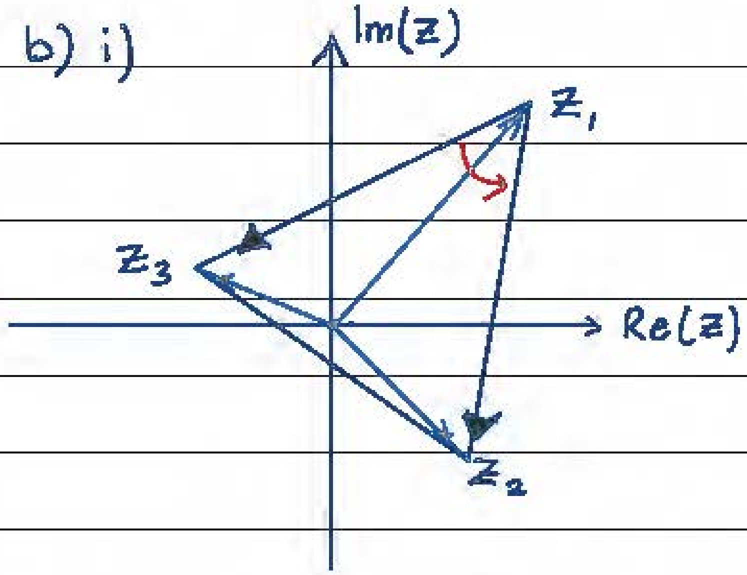
The algebraic manipulation caused problems for some students.

More reasons need to be given.

MARKER'S COMMENTS

Question 15

b) i)



$z_3 - z_1$

$z_3 - z_1$

Since the triangle is equilateral, each interior angle is $\frac{\pi}{3}$ and each side is equal.

We want to rotate vector $z_3 - z_1$ by $\pi/3$ to produce vector $z_2 - z_1$, which is the same magnitude as $z_3 - z_1$ — 1 mark for explanation

$$\therefore (z_3 - z_1) \times 1 \operatorname{cis} \frac{\pi}{3} = z_2 - z_1$$ — 1 mark

$$\therefore \operatorname{cis} \frac{\pi}{3} = \frac{z_2 - z_1}{z_3 - z_1}$$ 2

- $-\frac{1}{2}$ mark if there was no mention of the sides having the same magnitude

Note: Students need to clearly state that all interior angles were $\frac{\pi}{3}$ and all sides were equal.

Students must note that $(z_3 - z_1)\operatorname{cis}\frac{\pi}{3} = z_2 - z_1$ is an anti-clockwise rotation (mapping) of $z_3 - z_1$ onto $z_2 - z_1$ by $\frac{\pi}{3}$ (and both vectors have the same length)

Made with Goodnotes

MARKER'S COMMENTS

b) ii) From (i) $\frac{z_2 - z_1}{z_3 - z_1} = \text{cis}\frac{\pi}{3}$ ---- (1)

i.e $z_2 - z_1 = (z_3 - z_1)\text{cis}\frac{\pi}{3}$

Now similarly, $z_3 - z_2 = (z_1 - z_2)\text{cis}\frac{\pi}{3}$

i.e $z_1 - z_2$ is rotated in a clockwise direction by $\frac{\pi}{3}$ resulting in $z_3 - z_2$

$\therefore (z_1 - z_2)\text{cis}\frac{\pi}{3} = z_3 - z_2$

i.e $\text{cis}\frac{\pi}{3} = \frac{z_3 - z_2}{z_1 - z_2}$ ---- (2)

1 mark

(1) = (2)

$$\frac{z_3 - z_2}{z_1 - z_2} = \frac{z_2 - z_1}{z_3 - z_1}$$

$$(z_3 - z_2)(z_3 - z_1) = (z_2 - z_1)(z_1 - z_2)$$

$$z_3^2 - z_1 z_3 - z_2 z_3 + \cancel{z_2 z_1} = \cancel{z_2 z_1} - z_2^2 - z_1^2 + z_1 z_2$$

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_3 + z_1 z_2 + z_2 z_3$$

1 mark

2

Note: This part was not answered well.
Many incorrect statements were made in student's solutions.

MARKER'S COMMENTS

Question 15

c) i) Prove $\int_{-a}^{a} f(x)\,dx = \int_0^a [f(x)+f(-x)]\,dx$

Method 1

$$RHS = \int_0^a [f(x)+f(-x)]\,dx$$

$$= \int_0^a f(x)dx + \int_0^a f(-x)dx$$

Let $u=-x$

$\frac{du}{dx} = -1$

1 mk

$dx = -du$

When $x=a$, $u=-a$

$x=0$, $u=0$

$$= \int_0^a f(x)\,dx + \int_0^{-a} f(u) \times -du$$

$$= \int_0^a f(x)\,dx - \int_0^{-a} f(u)\,du$$

$$= \int_0^a f(x)\,dx + \int_{-a}^0 f(u)\,du$$

1 mark

by dummy variable

$$= \int_0^a f(x)\,dx + \int_{-a}^0 f(x)\,dx$$

$$= \int_{-a}^a f(x)\,dx$$

$= LHS$

2

MARKER'S COMMENTS

Question 15

c) i)

Method 2

$$\text{LHS} = \int_{-a}^{a} f(x)\,dx$$

$$= \int_{-a}^{0} f(x) + \int_{0}^{a} f(x)\,dx$$

Let $x = -u$ | $x=0,\ u=0$

$dx = -du$ | $x=-a,\ u=a$

$$= \int_{a}^{0} f(-u)\,-du + \int_{0}^{a} f(x)\,dx$$ — 1mk

$$= \int_{0}^{a} f(-u)\,du + \int_{0}^{a} f(x)\,dx$$

$$= \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx$$ (by dummy variable) — 1mk

$$= \int_{0}^{a} [f(x) + f(-x)]\,dx$$

$$= \text{RHS}$$

Made with Goodnotes

MARKER'S COMMENTS Question 15

c) i)

Method 3

Let $\int f(x)\,dx = F(x) + c$

$$LHS = \int_{-a}^{a} f(x)\,dx = \Big[F(x)\Big]_{-a}^{a}$$

$$= F(a) - F(-a)$$

$$RHS = \int_{0}^{a} f(x) + f(-x)\,dx$$

$$= \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(-x)\,dx$$

$$= \Big[F(x)\Big]_{0}^{a} - \Big[F(-x)\Big]_{0}^{a}$$

$$= F(a) - F(0) - \left[F(-a) - F(0)\right]$$

$$= F(a) - F(-a)$$

$$= LHS$$

$$\therefore \int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x) + f(-x)\,dx$$

Note: There are variety of methods available. Students mostly used the ones involving the dummy variable.

MARKER'S COMMENTS Question 15

c) ii)

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^x \sin^2 x}{1+e^x}\,dx$$

$$= \int_0^{\pi/2} \left[\frac{e^x \sin^2 x}{1+e^x} + \frac{e^{-x}\sin^2(-x)}{1+e^{-x}}\right] dx$$ — 1 mark for using (i)

$$= \int_0^{\pi/2} \frac{(1+e^{-x})e^x \sin^2 x + (1+e^x)e^{-x}\sin^2(-x)}{(1+e^x)(1+e^{-x})}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{e^x \sin^2 x + \sin^2 x + e^{-x}\sin^2(-x) + \sin^2(-x)}{1+e^{-x}+e^x+1}\,dx$$

$$= \int_0^{\pi/2} \frac{e^x \sin^2 x + \sin^2 x + e^{-x}\sin^2 x + \sin^2 x}{2+e^x+e^{-x}}\,dx$$

as $\sin^2(-x) = \sin x$

— 1 mark

$$= \int_0^{\pi/2} \frac{e^x \sin^2 x + 2\sin^2 x + e^{-x}\sin^2 x}{2+e^x+e^{-x}}\,dx$$

$$= \int_0^{\pi/2} \frac{\sin^2 x\,\cancel{(e^x + 2 + e^{-x})}}{\cancel{2+e^x+e^{-x}}}\,dx$$

$$= \int_0^{\pi/2} \sin^2 x\,dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1 - \cos 2x)\,dx$$

MARKER'S COMMENTS

Question 15

$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} - \frac{1}{2}\sin 2\left(\frac{\pi}{2}\right)\right) - (0-0)\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - \frac{1}{2}\sin \pi\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - 0\right)$$

$$= \frac{\pi}{4}$$

1 mark for evaluating

3

Note: • As part the second part of (c) you should assume that (c) i) has something to do with this question. Most students did.
Once applying part (c) i), those student who multiplied $\frac{e^{-x}\sin^2 x}{1+e^{-x}}$ by $\frac{e^x}{e^x}$ had more success. (See next page for alternative solution)

• When you replace $\sin^2(-x)$ with $\sin^2 x$ you should give a simple reason.
i.e $\sin(-x) = \sin(x)$ for all x.

MARKER'S COMMENTS Question 15

Method 2

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^x \sin^2 x}{1+e^x}\,dx$$

$$= \int_0^{\pi/2} \left[\frac{e^x \sin^2 x}{1+e^x} + \frac{e^{-x}\sin^2(-x)}{1+e^{-x}} \cdot \frac{e^x}{e^x}\right] dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{e^x \sin^2 x}{1+e^x} + \frac{\sin^2(-x)}{1+e^x}\right) dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{e^x \sin^2 x}{1+e^x} + \frac{\sin^2 x}{1+e^x}\right) dx \quad \text{as } \sin^2(-x) = \sin^2 x$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^2 x\,[\cancel{e^x + 1}]}{\cancel{1+e^x}}\,dx$$

$$= \int_0^{\pi/2} \sin^2 x\,dx$$

⋮

QUESTION 16

MARKER'S COMMENTS

a) $T_0 = 0$ $\quad T_n = \sqrt{T_{n-1}+2}$ $\quad$ Prove $T_n = 2\cos\left(\frac{\pi}{2^{n+1}}\right)$ $n = 0, 1, 2 \ldots$

Step 1 - Base case - Prove true for $n = 0$

Given $T_0 = 0$ $\quad T_n = 2\cos\left(\frac{\pi}{2^{n+1}}\right)$

$$T_0 = 2\cos\left(\frac{\pi}{2^{0+1}}\right)$$

$$= 2\cos\left(\frac{\pi}{2}\right)$$

1 mark prove true for base case

$$= 2\times 0$$

$$= 0$$

$\therefore$ True for $n=0$

Step 2 - Inductive Hypothesis - Assume True for $n=k$

Given $T_k = \sqrt{T_{k-1}+2}$, then $T_k = 2\cos\left(\frac{\pi}{2^{k+1}}\right)$, $k = 0, 1, 2, \cdots$

(ie assume proposition true for $T_0, T_1, T_2, \ldots, T_k$)

½ mark assumption

Step 3 - Inductive Step - Prove true for $n=k+1$

Prove given $T_{k+1} = \sqrt{T_k+2}$, then $T_{k+1} = 2\cos\left(\frac{\pi}{2^{k+2}}\right)$

Consider LHS = T_{k+1}

$$= \sqrt{T_k+2}$$

$$= \sqrt{2\cos\left(\frac{\pi}{2^{k+1}}\right)+2}$$ By the assumption.

1 mark using assumption

$$= \sqrt{2\left[\cos\left(\frac{\pi}{2^{k+1}}\right)+1\right]}$$

$$= \sqrt{2\left[\cos 2\left(\frac{\pi}{2\cdot 2^{k+1}}\right)+1\right]}$$

$$= \sqrt{2\left[\cos 2\left(\frac{\pi}{2^{k+2}}\right)+1\right]}$$

$\cos 2\theta = 2\cos^2\theta - 1$

$\therefore \cos 2\theta + 1 = 2\cos^2\theta$

Q16a) cont...

MARKER'S COMMENTS

$= \sqrt{2 \times 2\cos^2\left(\frac{\pi}{2^{k+2}}\right)}$

$= \sqrt{4\cos^2\left(\frac{\pi}{2^{k+2}}\right)}$

$= 2\cos\left(\frac{\pi}{2^{k+2}}\right)$

1 mark - use of $\cos 2\theta$ and proof.

$= RHS$

$\therefore LHS = RHS$

$\therefore$ true for $n=k+1$ if true for $n=k$

$\therefore$ By mathematical induction true for all integer $n \geq 0$.

$\frac{1}{2}$ mark. conclusion

/4

MARKER'S COMMENTS

Q16 cont...

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx \qquad n \geq 0$$

(i) $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$ and $I_{n-2} = \int_0^{\frac{\pi}{2}} \sin^{n-2} x \, dx$

$I_n = \int_0^{\frac{\pi}{2}} \sin^{n-1} x \cdot \sin x \, dx$ $\quad u = \sin^{n-1} x \quad v' = \sin x$

$u' = (n-1)\sin^{n-2} x (\cos x) \quad v = -\cos x$

$I_n = \left[-\cos x \sin^{n-1} x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2} x \cos x (-\cos x)$

$\frac{1}{2}$ mark to split $\sin^n x = \sin^{n-1} x \cdot \sin x$

$I_n = \left[-\cos\frac{\pi}{2} \sin^{n-1}\frac{\pi}{2} - 0\right] + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \, dx$

1 mark to correctly do integration by parts.

$I_n = \left[-\cos\frac{\pi}{2} \sin^{n-1}\frac{\pi}{2} - 0\right] + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \, (1 - \sin^2 x) \, dx$

$= (n-1)\left[\int_0^{\frac{\pi}{2}} \sin^{n-2} x - \sin^n x\right]$

1 mark to simplify I_n

$\frac{1}{2}$ mark to find I_{n-2} and establish result.

$I_n = (n-1)(I_{n-2} - I_n) \qquad I_n = nI_{n-2} - nI_n + I_{n-2} + I_n$

$I_n = nI_{n-2} - nI_n - I_{n-2} + I_n$

$nI_n = I_{n-2}(n-1)$

$$I_n = \frac{n-1}{n} I_{n-2}$$

MARKER'S COMMENTS

Q16 cont...

b) (ii)

1 mark

$$I_{2n} = \frac{2n-1}{2n} I_{2n-2}$$

$$= \frac{2n-1}{2n} \times \frac{2n-2-1}{2n-2} I_{2n-2-2}$$

$$= \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} I_{2n-4}$$

depletes an even number by 2 less than the time b4.

$$= \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times \frac{2n-5}{2n-4} I_{2n-6} \times \cdots I_4$$

1 mark establishing

$$= \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times \frac{2n-5}{2n-4} I_{2n-6} \times \cdots \underbrace{\frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}}$$

NOTE :

$$I_4 = \frac{3}{4} I_2$$

$$= \frac{3}{4} \times \frac{1}{2} I_0$$

$$= \frac{3}{4} \times \frac{1}{2} \times \int_0^{\pi/2} \sin^0 x \, dx$$ Must show this.

$$= \frac{3}{4} \times \frac{1}{2} \times \int_0^{\pi/2} 1 \, dx$$

$$= \frac{3}{4} \times \frac{1}{2} \times \Big[x\Big]_0^{\pi/2}$$

$$= \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

$$\therefore I_{2n} = \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times \frac{2n-5}{2n-4} \times \cdots \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

MARKER'S COMMENTS

Q16 continued

b) (ii)

$$I_{2n+1} = \frac{2n}{2n+1} I_{2n-1}$$

$$= \frac{2n}{2n+1} \times \frac{2n-2}{2n-1} I_{2n-3}$$

1 mark

$$= \frac{2n}{2n+1} \times \frac{2n-2}{2n-1} \times \frac{2n-4}{2n-3} I_{2n-5} \times \cdots \times I_5$$

$$= \frac{2n}{2n+1} \times \frac{2n-2}{2n-1} \times \frac{2n-4}{2n-3} I_{2n-5} \times \cdots \times \underbrace{\frac{4}{5} \times \frac{2}{3} \times 1}$$

$$I_5 = \frac{4}{5} I_3$$

1 mark

$$= \frac{4}{5} \times \frac{2}{3} I_1$$

Must establish

$$= \frac{4}{5} \times \frac{2}{3} \times \int_0^{\pi/2} \sin x \, dx$$

$$= \frac{4}{5} \times \frac{2}{3} \times \left[-\cos x\right]_0^{\pi/2}$$

$$= \frac{4}{5} \times \frac{2}{3} \times \left[-\cos\frac{\pi}{2} + \cos 0\right]$$

$$= \frac{4}{5} \times \frac{2}{3} \times (0+1)$$

$$= \frac{4}{5} \times \frac{2}{3} \times 1$$

$$\therefore I_{2n+1} = \frac{2n}{2n+1} \times \frac{2n-2}{2n-1} \times \frac{2n-4}{2n-3} \times \cdots \times \frac{4}{5} \times \frac{2}{3} \times 1$$

/4

MARKER'S COMMENTS

Q16 cont...

b) (iii) $I_k > I_{k+1}$

$I_{2n+1} > I_{2n+1+1}$ $\quad$ $I_{2n} > I_{2n+1}$

$\therefore I_{2n+1} > I_{2n+2}$

$$I_{2n+2} < I_{2n+1} < I_{2n}$$

$$\frac{2n+1}{2n+2} I_{2n} < I_{2n+1} < I_{2n}$$

$$\frac{2n+1}{2n+2} < \frac{I_{2n+1}}{I_{2n}} < 1$$

$$\frac{2n+1}{2n+2} < \frac{\frac{2n}{2n+1} \times \frac{2n-2}{2n-1} \times \frac{2n-4}{2n-3} \times \cdots \times \frac{4}{5} \times \frac{2}{3} \times 1}{\frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times \frac{2n-5}{2n-4} \times \cdots \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}} < 1$$

$$\frac{2n+1}{2n+2} < \frac{(2n)^2 \times (2n-2)^2 \times (2n-4)^2 \times \cdots \times 4^2 \times 2^2 \times \frac{2}{\pi}}{2n+1 \times (2n-1)^2 \times (2n-3)^2 \times (2n-5)^2 \times \cdots \times 3^2 \times 1^2} < 1$$

$$\frac{\pi}{2}\left(\frac{2n+1}{2n+2}\right) < \frac{2^2 \times 4^2 \times \cdots \times (2n-4)^2 \times (2n-2)^2 \times (2n)^2}{1 \times 3^2 \times 5^2 \times \cdots \times (2n-1)^2 (2n+1)} < \frac{\pi}{2}$$

$\frac{1}{2}$ mark to establish correct inequalities

$I_n = \frac{n-1}{n} I_{n-2}$

$\frac{1}{2}$ mark to divide by I_{2n}

1 mark to correctly establish final inequality.

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