QUESTION ONE (Start a new answer booklet)

Marks

3 (a) Solve the inequality $\dfrac{3x-8}{x} \geq 1$.

2 (b) Solve $\sin 2x = -\frac{1}{2}$, for $0 \leq x \leq 2\pi$.

2 (c) Show that $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan x\, dx = \frac{1}{2}\log 3$.

2 (d) Find $\displaystyle\int_0^{0\cdot 4} \frac{3\,dx}{4+25x^2}$.

3 (e) Simplify $\tan\left(2\cos^{-1}\frac{12}{13}\right)$, giving your answer as a fraction reduced to lowest terms.
HINT: Let $\alpha = \cos^{-1}\frac{12}{13}$.

QUESTION TWO (Start a new answer booklet)

Marks

3 (a) Use the substitution $x = 2\sin\theta$ to evaluate $\displaystyle\int_0^1 \sqrt{4-x^2}\,dx$.

9 (b) Consider the function $y = x\sqrt{2-x^2}$, with domain $-\sqrt{2} \leq x \leq \sqrt{2}$.

(i) Show that the function is odd.

(ii) Show that $\dfrac{dy}{dx} = \dfrac{2-2x^2}{\sqrt{2-x^2}}$.

(iii) Find the coordinates of any stationary points and then determine their nature.
You may use without proof the fact that $\dfrac{d^2y}{dx^2} = \dfrac{2x^3-6x}{(2-x^2)^{\frac{3}{2}}}$.

(iv) Notice that $\dfrac{dy}{dx} \to -\infty$ as $x \to \left(\sqrt{2}\right)^-$ (that is, as x approaches $\sqrt{2}$ from the left).
What does this fact tell you about the shape of the curve near $x = \sqrt{2}$?

(v) Draw a sketch of the curve, showing all x-intercepts, y-intercepts and stationary points.

(vi) Use the substitution $u = 2 - x^2$ to find the area between the curve and the x-axis from $x = 0$ to $x = \sqrt{2}$.

QUESTION THREE (Start a new answer booklet)

ks

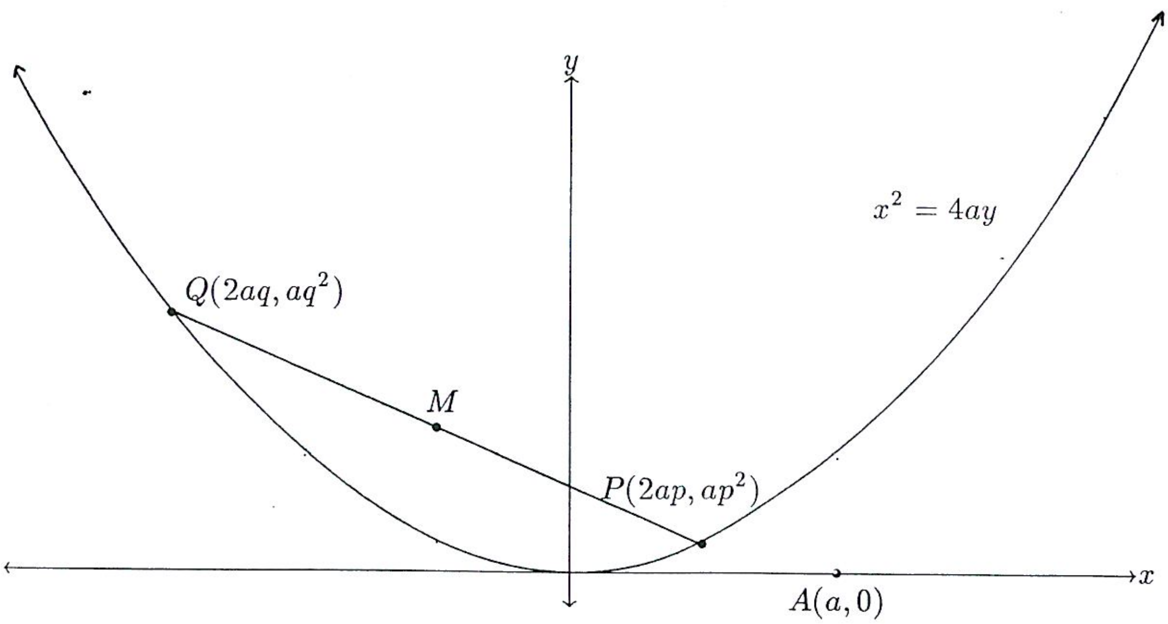
2 (a) Write out the expansion of $\left(3x^3 + \dfrac{1}{x}\right)^4$, giving the coefficients as integers.

2 (b) Sketch the graph of the function $y = |x - 5| - 2$, showing the coordinates of all significant points, including x-intercepts and y-intercepts.

2 (c)

The point M in the diagram above divides the interval AB externally in the ratio $1 : k$. Find k.

6 (d)



In the diagram above, the points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola with equation $x^2 = 4ay$.

(i) Write down the coordinates of the midpoint M of the chord PQ.

(ii) Show that the equation of the chord PQ is $y = \dfrac{(p+q)x}{2} - apq$.

(iii) Show that the condition for the chord PQ produced to pass through the point $A(a, 0)$ is:

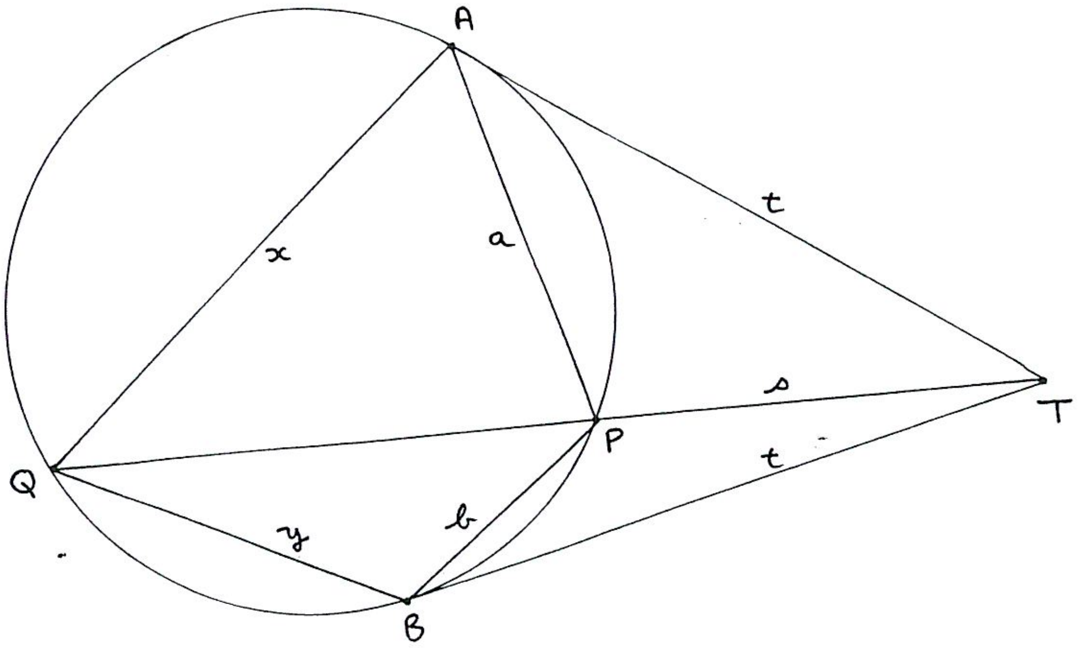
$$p + q = 2pq.$$

(iv) Find the cartesian equation of the locus of M, as the points P and Q move on the parabola subject to the constraint that PQ pass through $A(a, 0)$.

Exam continues overleaf ...

QUESTION FOUR (Start a new answer booklet)

(a)



The diagram above shows two tangents of equal length t drawn from a point T outside a circle, touching the circle at A and B. A secant through T meets the circle first at P and then at Q, where $TP = s$. The sides of the cyclic quadrilateral $APBQ$ are:

$$AP = a, \quad BP = b, \quad AQ = x, \quad BQ = y.$$

(i) Write a careful proof that $\triangle ATP \;|||\; \triangle QTA$.

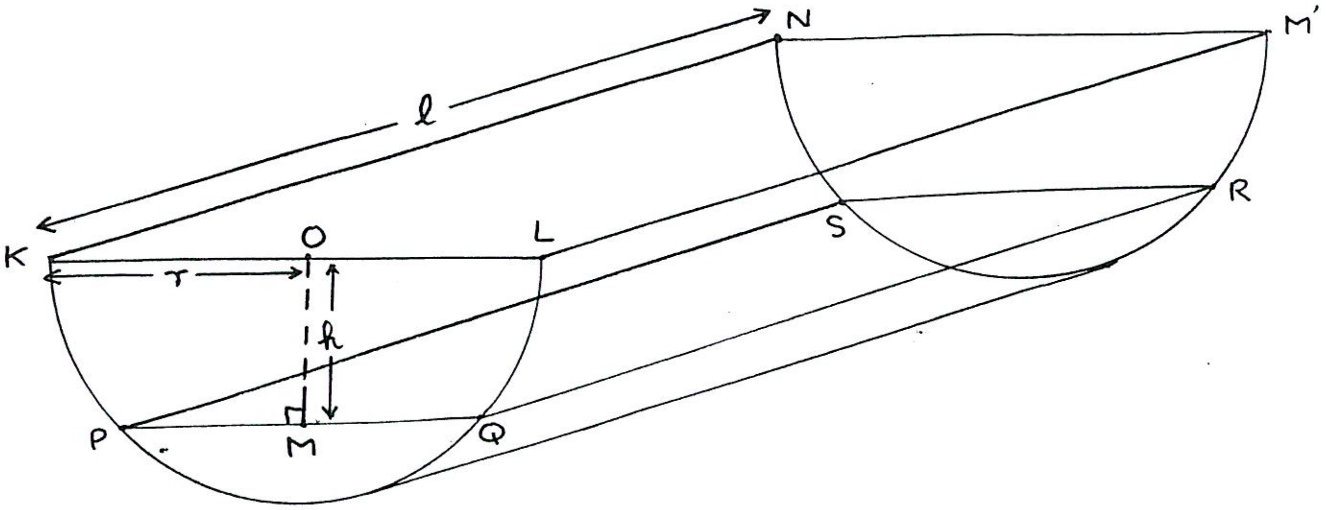
(ii) Hence express the ratio $\dfrac{a}{x}$ in terms of the lengths s and t.

(iii) Write down the corresponding expression for $\dfrac{b}{y}$, and hence prove that $ay = bx$.

(iv) Prove that $\dfrac{\text{area } \triangle APQ}{\text{area } \triangle BPQ} = \dfrac{a^2}{b^2}$.

QUESTION FOUR (Continued)

5 (b)



The diagram above shows a horse trough which is in the shape of half a cylinder, with length ℓ and radius r. It is partly filled with water, and the surface $PQRS$ of the water is a distance h below the top $KLM'N$ of the trough. Let M be the midpoint of PQ, and O be the midpoint of KL.

(i) Express the length PM in terms of h and r.

(ii) Hence show that the area A of the surface of the water is given by:

$$A = 2\ell\sqrt{r^2 - h^2}\,.$$

(iii) The water in the trough is evaporating in the hot Summer sun in such a way that the distance h from the top of the trough to the surface of the water is increasing at a constant rate.

The trough measurements are $\ell = 250$ cm and $r = 50$ cm, and the surface is descending at a constant 0·3 cm per day. Find the rate at which the surface area is decreasing when the surface of the water is 40 cm below the top of the trough (express your answer in units of cm^2 per day).

Exam continues overleaf ...

QUESTION FIVE (Start a new answer booklet)

4 (a) (i) Use long division to divide the polynomial $f(x) = x^4 - x^3 + x^2 - x + 1$ by the polynomial $d(x) = x^2 + 4$. Express your answer in the form:

$$f(x) = d(x)\,q(x) + r(x).$$

(ii) Hence find the values of the constants a and b so that $x^4 - x^3 + x^2 + ax + b$ is divisible by $x^2 + 4$.

5 (b) Consider the binomial expansion of $(3 + 11x)^{19}$.

(i) Let T_k be the kth term in the expansion (where the terms are written out in increasing powers of x). Show that:

$$\frac{T_{k+1}}{T_k} = \frac{11x(20-k)}{3k}.$$

(ii) Find the greatest coefficient in the expansion. Express your answer by giving the prime factorisation of this coefficient.

3 (c) Consider the binomial expansion:

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + \binom{n}{n}x^n.$$

(i) Use a suitable substitution to find the value of:

$$\binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + \cdots + 2^n\binom{n}{n}.$$

(ii) Differentiate both sides of the identity, and then use a suitable substitution to find the value of:

$$\binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} - \cdots + (-1)^{n-1}n\binom{n}{n}.$$

Exam continues next page . . .

QUESTION SIX (Start a new answer booklet)

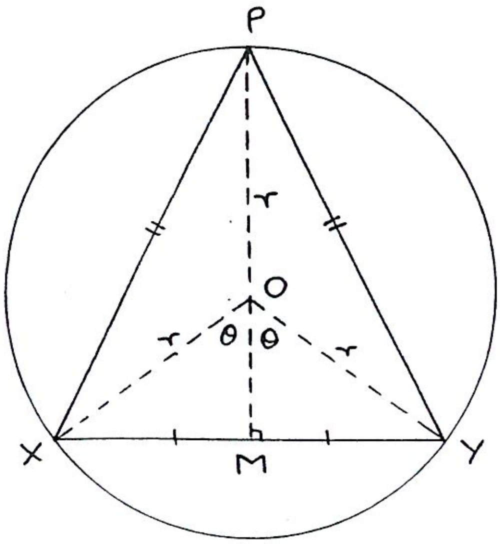
4 (a) The velocity v of a particle moving on the x-axis is given by:

$$v^2 = -3x^2 + 20x + 7.$$

Show that the particle is moving in simple harmonic motion, and find the centre, amplitude and period of that motion.

3 (b) Solve $2\cos^2\theta + \cos\theta = 1$, for $0 \leq \theta \leq \pi$.

5 (c)



In the diagram above, O is the centre of a circle of constant radius r. A variable chord XY subtends an angle 2θ at the centre O. Let P be the point on the major arc XY so that $\triangle XPY$ is isosceles with $XP = YP$. Let PO meet XY at M, so that PM is the perpendicular bisector of the chord XY.

(i) Prove that the area A of $\triangle XPY$ is given by:

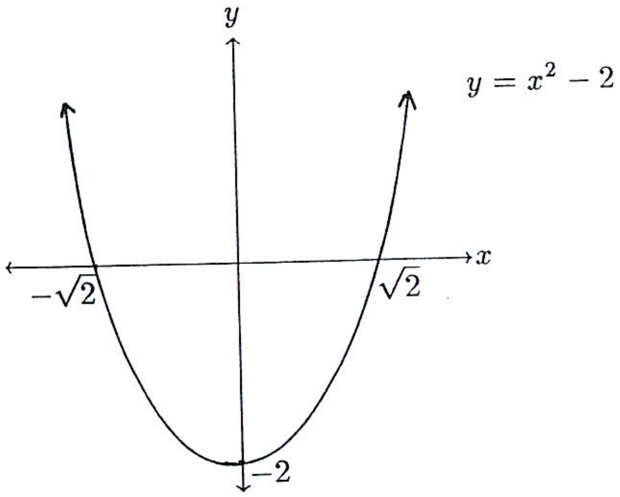
$$A = r^2 \sin\theta(1 + \cos\theta).$$

(ii) Show that $\dfrac{dA}{d\theta} = r^2(2\cos^2\theta + \cos\theta - 1)$, and find $\dfrac{d^2A}{d\theta^2}$.

(iii) Use part (b) above to show that $\triangle XPY$ has maximum area when it is an equilateral triangle.

QUESTION SEVEN (Start a new answer booklet)

(a)



It is desired to find approximations to $\sqrt{2}$ by applying Newton's method to find the roots of the function $f(x) = x^2 - 2$ sketched above. Let the initial value be x_1.

(i) State for which initial values x_1 the successive approximations given by Newton's method will eventually get close to $\sqrt{2}$, for which initial values they will eventually get close to $-\sqrt{2}$, and for which initial values the method will fail entirely.

(ii) Show that for the initial value x_1, the next approximation x_2 is:

$$x_2 = \frac{{x_1}^2 + 2}{2x_1}.$$

(iii) Using the initial value $x_1 = 2$, apply Newton's method three times, giving each successive approximation as a rational number without further rounding. Do not use decimal notation at all in this question.

4 (b) Use the method of mathematical induction to prove that for all integers $n \geq 2$, the expression $9^n - 8n - 1$ is divisible by 64.

3 (c) Consider the sum:

$$S_n = a + \frac{a+d}{r} + \frac{a+2d}{r^2} + \frac{a+3d}{r^3} + \frac{a+4d}{r^4} + \cdots + \frac{a+nd}{r^n},$$

where a, d and r are constants, with $|r| > 1$, and n is an integer.

(i) Write down the expression for rS_n.

(ii) By subtracting the expressions for S_n and for rS_n, find a formula for S_n.

(iii) Hence show that:

$$\lim_{n \to \infty} S_n = \frac{r(ar - a + d)}{(r-1)^2}.$$

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