Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Which of the following is the angle between the vectors $\underset{\sim}{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$?

A. 10°

B. 63°

C. 140°

D. 117°

$\underset{\sim}{a} \cdot \underset{\sim}{b} = -3 + 8 = |a||b| \cos\theta$

$\cancel{5} = \cancel{5} \times \sqrt{5} \cos\theta$

$\frac{1}{\sqrt{5}} = \cos\theta$

$= \theta$

2 Which of the following is equal to $\cos\theta$?

A. $\sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2}$

B. $2\cos^2\frac{\theta}{2} + 1$

C. $1 - 2\cos^2\theta$

D. $\dfrac{\sin\theta}{\tan\theta}$

$0 = 3x^2 - 2x - 5$

$0 = (3x - 5)(x + 1)$

3 The polynomial $P(x) = x^3 - x^2 - 5x - 3$ has a double root at $x = \alpha$.
What is the value of α?

~~A. $\frac{-5}{3}$~~

B. -1

C. 1

~~D. $\frac{5}{3}$~~

$2\alpha + \beta = 1$

$\beta = 1 - 2\alpha$

$2\alpha\beta + \alpha^2 + \cancel{\beta^2} = -5$

$2\alpha(1 - 2\alpha) + \alpha^2 = -5$

$2\alpha - 4\alpha^2 + \alpha^2 = -5$

$-3\alpha^2 + 2\alpha + 5 = 0$

$3\alpha^2 - 2\alpha - 5 = 0$

$(3\alpha - 5)(\alpha + 1) =$

4 Which of the following is equivalent to $\int \sin x \cos x \, dx$?

A. $-\cos 2x + c$

B. $-\frac{1}{2}\cos 2x + c$

C. $-\frac{1}{4}\cos 2x + c$

D. $\frac{1}{4}\cos 2x + c$

$= \int \frac{1}{2} \sin 2x \, dx$

$= \int -\frac{1}{4} \cos(2x)$

5 What is the domain of $y = 3\cos^{-1}(2x+1)$?

A. Domain $[-1, 0]$

B. Domain $(-1, 0)$

C. Domain $\left[-\frac{1}{2}, 0\right]$

D. Domain $\left[\frac{1}{2}, 0\right]$

7 7 7 8

6 In a Mathematics class a teacher can award one of 4 grades A, B, C or D to each student. What is the minimum number of students required so that at least 8 students are guaranteed to receive the same grade?

A. 28

B. 29

C. 32

D. 33

7 Which of the following is the range of the function $y = \frac{1}{x^2+1}$?

A. $(-\infty, \infty)$

B. $(-\infty, 1]$

C. $(0, 1]$

D. $[0, 1]$

8 A curve has an asymptote at $x = \frac{\pi}{3}$. Which of the following could be the equation of the curve?

A. $y = \sec\left(x - \frac{\pi}{3}\right)$

B. $y = \sec\left(x + \frac{\pi}{3}\right)$

C. $y = \cot\left(x - \frac{\pi}{3}\right)$

D. $y = \operatorname{cosec}\left(x + \frac{\pi}{3}\right)$

9 Which of the following is the primitive of $\frac{4}{\sqrt{9 - x^2}}$?

A. $\frac{4}{3}\sin^{-1}\frac{x}{3} + c$

B. $\frac{4}{3}\sin^{-1}3x + c$

C. $4\sin^{-1}3x + c$

D. $4\sin^{-1}\frac{x}{3} + c$

$4\left(\frac{1}{\sqrt{9-x^2}}\right)$

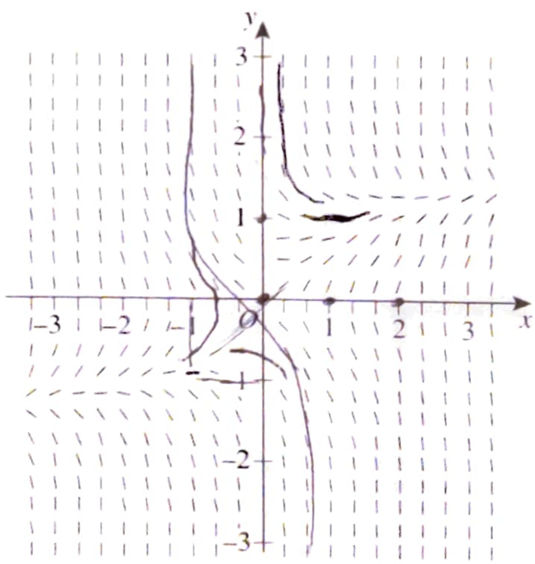
10 Which of the following best represents the differential equation shown in the slope field?

A. $\frac{dy}{dx} = \frac{x}{y} - y^2$

B. $\frac{dy}{dx} = \frac{x}{y} + y^2$

C. $\frac{dy}{dx} = -\frac{x}{y} - y^2$

D. $\frac{dy}{dx} = -\frac{x}{y} + y^2$



$\frac{dy}{dx}\left(\frac{1}{y}\right) = \frac{x}{y^2} - y$

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Solve $\dfrac{3}{2x-5} \leq -1$. **3**

(b) Find the derivative of $\cos^{-1}\left(\dfrac{3x}{2}\right)$. **1**

(c) For what values of m are the two distinct vectors $\begin{pmatrix} m \\ 2m+6 \end{pmatrix}$ and $\begin{pmatrix} m+1 \\ -1 \end{pmatrix}$ perpendicular? **2**

(d) Find $\displaystyle\int \cos^2 4x \, dx$. **2**

(e) Evaluate $\displaystyle\int_0^{\frac{\pi}{6}} \cos x \sin^3 x \, dx$. **3**

(f) For what values of m is the polynomial $x^2 - (m+2)x + 2(m+2)$ positive for all values of x? **2**

(g) Consider the expansion $(2x-p)^9$. The coefficient of x^6 is -344064. Find the value of p. **2**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Find $\int \frac{dx}{25+4x^2}$. **2**

(b) Given that $\sin\theta = \frac{3}{4}$ and $\frac{\pi}{2} \le \theta \le \pi$ determine the exact value of $\tan 2\theta$. **3**

(c) (i) Find the derivative of $x\log_e x - x$. **1**

(ii) Hence evaluate $\int_{\sqrt{e}}^{e} \log_e x \, dx$. Leave your answer in exact form. **2**

(d) A mixed volleyball team of eight players is selected from ten males and nine females. In how many ways can this be done if the team must have at least 2 female players? **2**

(e) The vectors $\underset{\sim}{u} = \begin{pmatrix} a \\ 3 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$ are parallel. Find the value of a. **2**

(f) The polynomial $P(x) = (x-p)^3 + q$ is zero at $x = 1$ and when divided by x the remainder is -7. Find the possible values of p. **3**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet

(a) Evaluate $\int_{-1}^{3} x\sqrt{3-x}\,dx$ using the substitution $u = 3 - x$. **4**

(b) (i) Express $\sqrt{3}\cos x + \sin x$ in the form $R\cos(x-\alpha)$ where $R > 0$ and α is acute. **2**

(ii) Hence solve $\sqrt{3}\cos x + \sin x = 1$ for $0 \le x \le 2\pi$. **2**

(c) Use the process of Mathematical induction to prove **3**

$2\times1! + 5\times2! + 10\times3! + \ldots + (n^2+1)n! = n(n+1)!$ for all positive integers n.

(d) A population of mice in a meadow after t years satisfies the logistic **4**

differential equation $\frac{dP}{dt} = \frac{3P}{2500}(2500-P)$, where the initial population of

mice is 500.

Given $\frac{1}{P} + \frac{1}{2500-P} = \frac{2500}{P(2500-P)}$, solve the differential equation to find the

population P of the mice at time t. Express your answer in the form

$P = \frac{A}{1-Be^{-kt}}$ where A, B and k are integers.

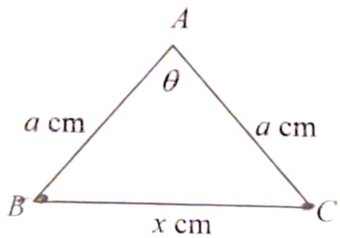
End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Solve the differential equation $(x^2+1)\dfrac{dy}{dx} = 6xy$ where $x=1$ and $y=2$ giving your answer as y in terms of x. **3**

(b) Given that $y = e^{2x} + e^{-2x}$, determine the values of constants a and b that satisfy the following differential equation $\dfrac{d^2y}{dx^2} + a\dfrac{dy}{dx} + by = 5e^{2x} + e^{-2x}$. **4**

(c) In the triangle ABC, $AB = AC = a$ cm. The angle BAC is increasing at the rate of 2 radians/min. Let $\angle BAC = \theta$ radians and $BC = x$ cm.



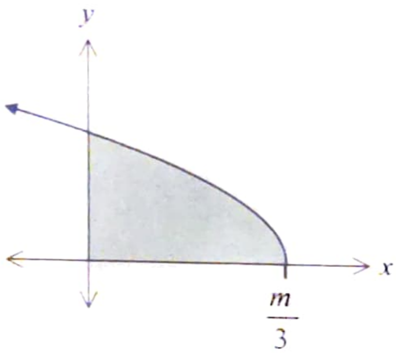
NOT TO SCALE

(i) Show that $x = a\sqrt{2-2\cos\theta}$. $x = a\sqrt{2-2\cos\theta}$ **1**

(ii) Show that $\dfrac{dx}{d\theta} = \dfrac{a\sin\theta}{\sqrt{2-2\cos\theta}}$. **1**

(iii) Determine, in terms of a, the rate with respect to time at which BC is increasing when $\theta = \dfrac{\pi}{3}$ radians. **2**

(d) Let $f(x) = \sqrt{m-3x}$ for $x < \dfrac{m}{3}$. The graph of $y = f(x)$ is shown. **4**



The area enclosed by the graph $y = f(x)$, the x-axis and the y-axis is rotated about the y axis. Find the value of m such that the volume of the solid formed is $\dfrac{5000\pi}{27}$ cubic units.

End of Paper

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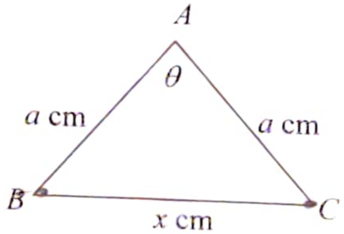

NOT TO SCALE

(i) Show that $x=a\sqrt{2-2\cos\theta}$. $x = a\sqrt{2-2\cos\theta}$ **1**

(ii) Show that $\dfrac{dx}{d\theta}=\dfrac{a\sin\theta}{\sqrt{2-2\cos\theta}}$. **1**

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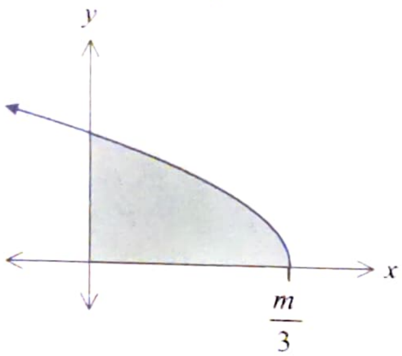


The area enclosed by the graph $y=f(x)$, the x-axis and the y-axis is rotated about the y axis. Find the value of m such that the volume of the solid formed is $\dfrac{5000\pi}{27}$ cubic units.

End of Paper

MATHEMATICS EXTENSION 1 TRIAL 2022

1. $\underset{\sim}{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ $\underset{\sim}{b} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$

$\cos\theta = \dfrac{a \cdot b}{|a||b|}$

$= \dfrac{(3 \times -1 + 4 \times 2)}{\sqrt{9+16}\sqrt{1+4}}$

$= \dfrac{5}{5\sqrt{5}} = \dfrac{1}{\sqrt{5}}$

$\theta = \cos^{-1}\dfrac{1}{\sqrt{5}}$

$\fallingdotseq 63^\circ$ (B)

2. (D) $\dfrac{\sin\theta}{\tan\theta} = \dfrac{\sin\theta}{\frac{\sin\theta}{\cos\theta}}$

$= \sin\theta \times \dfrac{\cos\theta}{\sin\theta}$

$= \cos\theta$

3. $P(x) = x^3 - x^2 - 5x - 3$

$P'(x) = 3x^2 - 2x - 5$

Double root $\therefore P'(a) = 0$

$0 = 3a^2 - 2a - 5$ $\quad \left[\begin{array}{l} P\ -15 \\ S\ -2 \\ F\ -5, 3 \end{array} \right.$

$0 = 3a^2 + 3a - 5a - 5$

$0 = 3a(a+1) - 5(a+1)$

$0 = (a+1)(3a-5$

$\therefore a = -1$ or $\frac{5}{3}$

$P(-1) = (-1)^3 - (-1)^2 - 5(-1) - 3$

$= -1 - 1 + 5 - 3$

$= 0$

$\therefore P(-1) = 0 \quad P'(-1) = 0$

$\therefore$ (B)

4. $\displaystyle\int \sin x \cos x \, dx = \frac{1}{2}\int \sin 2x \, dx$

$= \frac{1}{2}\left(-\dfrac{\cos 2x}{2}\right) + c$

$= -\dfrac{\cos 2x}{4} + c$

(C)

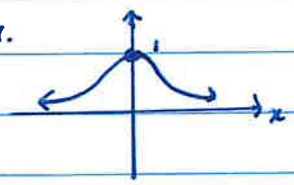
5. $-1 \leq 2x + 1 \leq 1$

$-2 \leq 2x \leq 0$

$-1 \leq x \leq 0$ $\quad [-1, 0]$ $\quad$ (A)

6. $4 \times 7 + 1 = 29$ $\quad$ (B)

7.



$0 < y \leq 1$

$(0, 1]$ $\quad$ (C)

8. A. $y = \sec 0 = \dfrac{1}{\cos 0} = 1$

B. $y = \sec\left(\frac{2\pi}{3}\right)$

C. $y = \cot(0)$ $\quad \dfrac{\cos 0}{\sin 0}$ $\quad \sin 0 = 0$ $\therefore$ (C)

undefined at $x = 0$

9. $\displaystyle\int \frac{4}{\sqrt{3^2 - x^2}} \, dx = 4\sin^{-1}\frac{x}{3} + c$ $\quad$ (D)

10. (A) $x = 1$ $y = 1$ $\dfrac{dy}{dx} = 1 - 1 = 0$ ✓ $\quad$ (A or D)

$x = -2$ $y = -1$ $\dfrac{dy}{dx} = 1$ ✓

D $\dfrac{dy}{dx} = -2 + 1$ ✗

$= -1$

$\therefore$ (A)

1	B
2	D
3	B
4	C
5	A
6	B
7	C
8	C
9	D
10	A.

(15 Marks)

11 a) $\dfrac{3}{2x-5} \leq -1$

critical values

$x \neq \dfrac{5}{2}$

$\dfrac{3}{2x-5} = -1$

$3 = -2x+5$

$2x = 2$

$x = 1$

test $x = 2$ $\quad \dfrac{3}{-1} \leq -1$ ✓

$1 \leq x < 2\frac{1}{2}$ [3]

b) $y = \cos^{-1}\dfrac{3x}{2}$

$\dfrac{dy}{dx} = \dfrac{-3}{\sqrt{4-9x^2}}$

$a = 2$
$f(x) = 3x$
$f'(x) = 3$

[1]

c) perpendicular when

$\begin{pmatrix} m \\ 2m+6 \end{pmatrix} \cdot \begin{pmatrix} m+1 \\ -1 \end{pmatrix} = 0$

$m(m+1) + -1(2m+6) = 0$

$m^2 + m - 2m - 6 = 0$

$m^2 - m - 6 = 0$

$(m-3)(m+2) = 0$

$\therefore m = 3$ or -2 [2]

d) $\displaystyle\int \cos^2 4x \, dx = \int \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 8x\right) dx$

$= \frac{1}{2}x + \frac{1}{2}\frac{\sin 8x}{8} + c$

$= \frac{1}{2}x + \frac{1}{16}\sin 8x + c$ [2]

e) $\displaystyle\int_0^{\frac{\pi}{6}} \cos x \sin^3 x \, dx$

$= \left[\dfrac{\sin^4 x}{4}\right]_0^{\frac{\pi}{6}}$

$= \dfrac{(\sin\frac{\pi}{6})^4}{4} - \dfrac{\sin^4 0}{4}$

$= \left(\frac{1}{4} \times \frac{1}{2^4}\right) - 0$

$= \dfrac{1}{64}$ [3]

f) For positive $\Delta < 0$

$\Delta = b^2 - 4ac$

$= [-(m+2)]^2 - 4(2(m+2))$

$= m^2 + 4m + 4 - 8m - 16$

$= m^2 - 4m - 12$

$= (m-6)(m+2)$ $\quad m = -2, 6$

$(m-6)(m+2) < 0$ $\quad$ test $m = 0$

$-6 \times 2 < 0$ ✓

$-2 < m < 6$ [2]

g) coefficient of x^6 $= {}^9C_6 (2x)^6 (-p)^3$

$-344\,064 = -5376p^3$

$64 = p^3$

$p = 4$ [2]

(15 Marks)

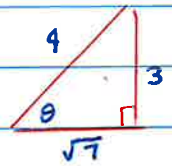
12 a) $\int \frac{dx}{25+4x^2} = \frac{1}{2}\int \frac{2\,dx}{5^2+(2x)^2}$

$a=5$
$f(x)=2x$
$f'(x)=2$

$= \frac{1}{5} \times \frac{1}{2} \tan^{-1}\frac{2x}{5} + c$

$= \frac{1}{10} \tan^{-1}\frac{2x}{5} + c$ [2]

b) $\sin\theta = \frac{3}{4}$ (Q2)



$\tan\theta = \frac{-3}{\sqrt{7}}$

$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

$= \frac{2 \times \frac{-3}{\sqrt{7}}}{1-\left(\frac{-3}{\sqrt{7}}\right)^2}$

$= \frac{\frac{-6}{\sqrt{7}}}{1-\frac{9}{7}}$

$= \frac{-\cancel{6}^{\,3}}{\sqrt{7}} \times \frac{-7}{\cancel{2}}$

$= \frac{21}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}}$

$= \frac{21\sqrt{7}}{7}$

$= 3\sqrt{7}$ [3]

c)

(i) $y = x\log_e x - x$

$u = x$
$u' = 1$
$v = \log_e x$
$v' = \frac{1}{x}$

$\frac{dy}{dx} = x \times \frac{1}{x} + \log_e x \times 1 - 1$

$= 1 + \log_e x - 1$

$= \log_e x$ [1]

(ii) $\int_{\sqrt{e}}^{e} \log_e x\,dx = \left[x\log_e x - x\right]_{\sqrt{e}}^{e}$

$= (e\log_e e - e) - (\sqrt{e}\log_e\sqrt{e} - \sqrt{e})$

$= (e-e) - (\sqrt{e}\log_e e^{\frac{1}{2}} - \sqrt{e})$

$= 0 - (\frac{1}{2}\sqrt{e}\log_e e - \sqrt{e})$

$= 0 - \frac{1}{2}\sqrt{e} + \sqrt{e}$

$= \frac{1}{2}\sqrt{e}$ [2]

d) 8 players 10 males 9 females

At least 2 female = Total − (0F + 1F)

$= {}^{19}C_8 - \left[{}^{9}C_0 \times {}^{10}C_8 + {}^{9}C_1 \times {}^{10}C_7\right]$

$= 74\,457$ [2]

(15 Marks)

2)(e) $\underset{\sim}{u} = \begin{pmatrix} a \\ 3 \end{pmatrix}$ $\underset{\sim}{v} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$

For parallel $\begin{pmatrix} a \\ 3 \end{pmatrix} = k\begin{pmatrix} 2 \\ 7 \end{pmatrix}$

$3 = k \times 7$

$\frac{3}{7} = k$

$\therefore\ a = \frac{3}{7} \times 2$

$\underline{a = \frac{6}{7}}$ [2]

(f) $P(x) = (x-p)^3 + q$

$\left.\begin{array}{l} P(1) = 0 \\ P(0) = -7 \end{array}\right\}$

$P(1) = (1-p)^3 + q$

$0 = (1-p)^3 + q$ (1)

$P(0) = (-p)^3 + q$

$-7 = (-p)^3 + q$

$-7 = -p^3 + q$

$-7 + p^3 = q$ (2)

Sub (2) into (1)

$0 = (1-p)^3 - 7 + p^3$

$0 = 1 - 3p + 3p^2 \cancel{-p^3} - 7 \cancel{+p^3}$

$0 = 3p^2 - 3p - 6$

$0 = p^2 - p - 2$

$0 = (p-2)(p+1)$

$\therefore\ \underline{p = 2 \text{ or } -1}$

[3]

(13) $\displaystyle\int_{-1}^{3} x\sqrt{3-x}\,dx$

$u = 3 - x$

$\frac{du}{dx} = -1$

$-du = dx$

$x = 3 \quad u = 0$

$x = -1 \quad u = 4$

$x = 3 - u$

$= -\displaystyle\int_{4}^{0} (3-u)\,u^{\frac{1}{2}}\,du$

$= \displaystyle\int_{0}^{4} \left(3u^{\frac{1}{2}} - u^{3/2}\right) du$

$= \left[\dfrac{3u^{3/2}}{\frac{3}{2}} - \dfrac{u^{5/2}}{\frac{5}{2}}\right]_0^4$

$= \left[2u^{3/2} - \frac{2}{5}u^{\frac{5}{2}}\right]_0^4$

$= \left[2(4)^{3/2} - \frac{2}{5}(4)^{5/2}\right] - [0 - 0]$

$= 16 - \frac{64}{5}$

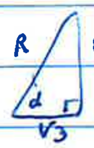
$= \underline{\frac{16}{5}}$

[4]

b)(i) $\sqrt{3}\cos x + \sin x = R\cos x\cos\alpha + R\sin x\sin\alpha$

$R\cos\alpha = \sqrt{3}$ $\quad R\sin\alpha = 1$

$\cos\alpha = \frac{\sqrt{3}}{R}$ $\quad \sin\alpha = \frac{1}{R}$



$R = \sqrt{3+1}$

$= 2$

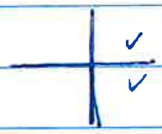
$\tan\alpha = \frac{1}{\sqrt{3}}$

$\alpha = \frac{\pi}{6}$

$\therefore\ \underline{\sqrt{3}\cos x + \sin x = 2\cos\left(x - \frac{\pi}{6}\right)}$ [2]

(ii) $2\cos\left(x - \frac{\pi}{6}\right) = 1$

$\cos\left(x - \frac{\pi}{6}\right) = \frac{1}{2}$



rel'd∠ $= \frac{\pi}{3}$

$\therefore\ x - \frac{\pi}{6} = \frac{\pi}{3}$ or $x - \frac{\pi}{6} = \frac{5\pi}{3}$

$x = \frac{\pi}{3} + \frac{\pi}{6}$ $\quad x = \frac{5\pi}{3} + \frac{\pi}{6}$

$= \underline{\frac{\pi}{2}}$ or $= \underline{\frac{11\pi}{6}}$ [2]

(c) step 1 : Prove true for $n=1$

$LHS = 2\times 1!$ $\qquad$ $RHS = 1(1+1)!$

$= 2$ $\qquad$ $= 2!$ $\qquad$ $\therefore LHS = RHS$

$= 2$ $\qquad$ True for $n=1$

step 2: Assume true for $n=k$

$2\times 1! + 5\times 2! + 10\times 3! + \dots + (k^2+1)k! = k(k+1)!$ $\qquad *$

step 3: Prove true for $n=k+1$

RTP: $\underbrace{2\times 1! + 5\times 2! + 10\times 3! + \dots + (k^2+1)k!}_{S_k} + [(k+1)^2+1](k+1)! = (k+1)(k+2)!$

$LHS = k(k+1)! + [k^2+2k+1+1](k+1)!$ (by assumption $*$)

$= (k+1)![k + k^2+2k+2]$

$= (k+1)! [k^2+3k+2]$

$= \underbrace{(k+1)!(k+2)}(k+1)$

$= (k+2)!(k+1)$

$= RHS$

step 4: Since true for $n=1$, then true for $n=2$, $n=3$ and so on for all positive integers.

[3]

d) $\frac{dP}{dt} = \frac{3P}{2500}(2500-P)$ Trivial solns: $P=0$

$P = 2500$

$$\frac{dP}{\frac{P}{2500}(2500-P)} = 3\,dt$$

$$\int \frac{2500}{P(2500-P)}\,dP = \int 3\,dt$$

$$\int \left(\frac{1}{P} + \frac{1}{2500-P}\right) dP = 3\int dt$$

$$\ln|P| - \ln|2500-P| = 3t + c$$

$$\ln\left|\frac{P}{2500-P}\right| = 3t + c$$

$$\left|\frac{P}{2500-P}\right| = e^{3t+c}$$

$$= e^{3t} \times e^{c}$$

$$= Ae^{3t}$$ where $A = e^{c}$

$$\frac{P}{2500-P} = Ae^{3t}$$ $A = \pm e^{c}$

when $t=0$ $\frac{500}{2000} = Ae^{0}$

$P = 500$

$$\frac{1}{4} = A$$

$$\therefore \frac{P}{2500-P} = \frac{1}{4}e^{3t}$$

$$4P = e^{3t}(2500-P)$$

$$4P = 2500e^{3t} - Pe^{3t}$$

$$4P + Pe^{3t} = 2500e^{3t}$$

$$P(4+e^{3t}) = 2500e^{3t}$$

$$P = \frac{2500e^{3t}}{4+e^{3t}}$$ $\div e^{3t}$

$$= \frac{2500}{\frac{4}{e^{3t}} + 1}$$

$$= \frac{2500}{1+4e^{-3t}}$$ $B = -4$ [4]

or $P = \frac{2500}{1-(-4)e^{-3t}}$

(15 Marks)

14 $(x^2+1)\frac{dy}{dx} = 6xy$

a) $\int \frac{dy}{y} = \int \frac{6x}{x^2+1}\,dx$

$\ln|y| = 3\int \frac{2x}{x^2+1}\,dx$

$\ln|y| = 3\ln|x^2+1| + c$

$x=1 \quad y=2 \qquad \ln 2 = 3\ln 2 + c$

$-2\ln 2 = c$

$\therefore \ln y = 3\ln(x^2+1) - 2\ln 2$

$\ln y = \ln(x^2+1)^3 - \ln 2^2$

$y = \frac{(x^2+1)^3}{4}$

[3]

b) $y = e^{2x} + e^{-2x}$

$\frac{dy}{dx} = 2e^{2x} - 2e^{-2x}$

$\frac{d^2y}{dx^2} = 4e^{2x} + 4e^{-2x}$

$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 4e^{2x} + 4e^{-2x} + 2ae^{2x} - 2ae^{-2x} + be^{2x} + be^{-2x}$

$= (4+2a+b)e^{2x} + (4-2a+b)e^{-2x}$

Equating coefficients: $4+2a+b = 5$ $\qquad 4-2a+b = 1$

$2a+b = 1$ ① $\qquad -2a+b = -3$ ②

$+ \quad -2a+b = -3$ ② $\qquad -2a-1 = -3$

$2b = -2 \qquad -2a = -2$

$b = -1 \qquad a = 1$

$\therefore a = 1 \quad b = -1$

[4]

(c) cosine rule:

$c^2 = a^2 + b^2 - 2ab\cos c$

(i) $x^2 = a^2 + a^2 - 2a^2\cos\theta$

$= 2a^2 - 2a^2\cos\theta$ (factorise a^2)

$x = a\sqrt{2 - 2\cos\theta}$ which is required [1]

ii. $\frac{dx}{d\theta} = a \times \frac{1}{\cancel{2}}(2 - 2\cos\theta)^{-\frac{1}{2}} \times \cancel{2}\sin\theta$

$= \frac{a\sin\theta}{\sqrt{2 - 2\cos\theta}}$ which is required [1]

(iii) $\frac{dx}{dt} = \frac{dx}{d\theta} \times \frac{d\theta}{dt}$

$= \frac{a\sin\theta}{\sqrt{2-2\cos\theta}} \times 2$ when $\theta = \frac{\pi}{3}$

$= \frac{2a\sin\frac{\pi}{3}}{\sqrt{2 - 2\cos\frac{\pi}{3}}}$

$= \frac{2a\frac{\sqrt{3}}{2}}{\sqrt{1}}$

$= \sqrt{3}a$ cm/min [2]

d) $V = \pi\int x^2 dy$

$= \frac{\pi}{9}\int_0^{\sqrt{m}} (y^4 - 2my^2 + m^2)\, dy$

$= \frac{\pi}{9}\left[\frac{y^5}{5} - 2m\frac{y^3}{3} + m^2 y\right]_0^{\sqrt{m}}$

$= \frac{\pi}{9}\left[\frac{(\sqrt{m})^5}{5} - \frac{2m}{3}(\sqrt{m})^3 + m^2\sqrt{m}\right]$

$= \frac{\pi}{9}\left[\frac{m^2\sqrt{m}}{5} - \frac{2}{3}m^2\sqrt{m} + m^2\sqrt{m}\right]$

$= \frac{\pi}{9} \times m^2\sqrt{m}\left(\frac{1}{5} - \frac{2}{3} + 1\right)$

$= m^2\sqrt{m}\,\frac{\pi}{9} \times \frac{8}{15}$

$= \frac{8}{135}m^2\sqrt{m}\,\pi$

$y = \sqrt{m - 3x}$ $x = 0$ $y = \sqrt{m}$

$y^2 = m - 3x$

$3x = m - y^2$

$x = \frac{m - y^2}{3}$

$x^2 = \frac{(m - y^2)^2}{9}$

$= \frac{m^2 - 2my^2 + y^4}{9}$

$\frac{8}{135} \cdot m^{5/2}\pi = \frac{5000\pi}{27}$

$m^{5/2} = 3125$

$m = 25$ [4]