Sydney Girls High School
2024
Trial Higher School Certificate
Examination

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using a black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- In Questions 11-14, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 60 marks

- Attempt Questions 11-14
- Allow about 1 hour and 45 minutes for this section

Name: …………………………………………………………	**THIS IS A TRIAL PAPER ONLY**
Teacher: …………………………………………………………	It does not necessarily reflect the format or the content of the 2024 HSC Examination Paper in this subject.

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10.

Questions **Marks**

1 What is the magnitude of $2\underset{\sim}{i} + 3\underset{\sim}{j}$? **1**

A. 5

B. $\sqrt{5}$

C. 13

D. $\sqrt{13}$

2 What is the value of $\displaystyle\int_0^{\frac{3}{2}} \frac{1}{\sqrt{9-x^2}}\,dx$? **1**

A. $\dfrac{\pi}{6}$

B. $\dfrac{\pi}{4}$

C. $\dfrac{\pi}{3}$

D. $\dfrac{\pi}{2}$

3 Which of the following is equivalent to $\sin x - \sqrt{3}\cos x$? **1**

A. $2\sin\left(x - \dfrac{\pi}{3}\right)$

B. $2\sin\left(x + \dfrac{\pi}{3}\right)$

C. $2\sin\left(x - \dfrac{\pi}{6}\right)$

D. $2\sin\left(x + \dfrac{\pi}{6}\right)$

4 The parametric equations of a curve are $x = 1 + t$ and $y = \dfrac{1-t}{1+t}$. **1**

What is the Cartesian equation of the curve?

A. $y = \dfrac{2}{x} - 1$

B. $y = \dfrac{1}{x} - 2$

C. $y = \dfrac{1}{x} + 2$

D. $y = \dfrac{2}{x} + 1$

5 What is the number of solutions to the equation $e^x + e^{-x} = x + 2$? **1**

A. 0

B. 1

C. 2

D. 3

6 What is the primitive of $(2\cos^2 3x - 1)^2$? **1**

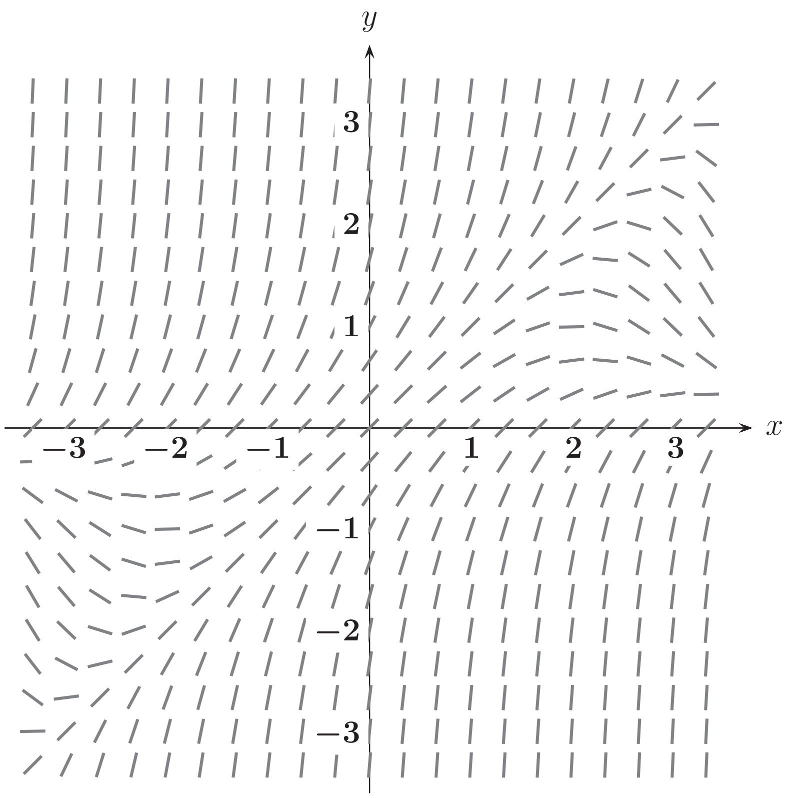
A. $\frac{1}{24}\cos 12x + \frac{1}{2}x + C$

B. $\frac{1}{24}\sin 12x + \frac{1}{2}x + C$

C. $-\frac{1}{24}\cos 12x + \frac{1}{2}x + C$

D. $-\frac{1}{24}\sin 12x + \frac{1}{2}x + C$

7 What is the differential equation that best matches the direction field shown? **1**



A. $\frac{dy}{dx} = x^2 - xy + 1$

B. $\frac{dy}{dx} = \frac{2xy}{1+x^2} + 1$

C. $\frac{dy}{dx} = y^2 - xy + 1$

D. $\frac{dy}{dx} = \frac{2xy}{1+y^2} + 1$

8 Which of the following is the value of $\cos^{-1}(\cos a)$ given that $\pi < a < \dfrac{3\pi}{2}$? **1**

A. a

B. $\pi - a$

C. $2\pi - a$

D. $3\pi - a$

9 Let θ be the angle between $\underset{\sim}{u}$ and $\underset{\sim}{v}$, where $\underset{\sim}{u} \cdot \underset{\sim}{v} \neq 0$. **1**

What does the following expression simplify to?

$$\frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{\left(\text{proj}_{\underset{\sim}{v}}\underset{\sim}{u}\right) \cdot \left(\text{proj}_{\underset{\sim}{u}}\underset{\sim}{v}\right)}$$

A. $\sin^2 \theta$

B. $\text{cosec}^2 \theta$

C. $\cos^2 \theta$

D. $\sec^2 \theta$

10 A box contains 400 balls, each of which is blue, red, green, yellow or orange. The ratio of blue to red to green balls is $1 : 4 : 2$. The ratio of green to yellow to orange balls is $1 : 3 : 6$. **1**

What is the minimum number of balls that must be drawn to ensure that at least 50 balls of one colour are selected?

A. 195

B. 196

C. 197

D. 198

Examination continues overleaf...

Section II

60 marks
Attempt Questions 11-14
Allow about 1 hour and 45 minutes for this section

Answer each question on the writing paper supplied. Start each question on a NEW page. Extra writing paper are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start on a NEW page **Marks**

(a) Differentiate $\left(\sin^{-1} x\right)^3$. **2**

(b) Solve $\dfrac{3}{x-2} \leq 1$. **3**

(c) Use the substitution $u = e^x$ to find $\displaystyle\int \frac{e^x}{9+4e^{2x}}\,dx$. **3**

(d) Consider the function $f(x) = \dfrac{3}{x-2} - 5$ for $x > 2$.

(i) State the range of $f^{-1}(x)$. **1**

(ii) Find the inverse function $f^{-1}(x)$. **3**

(iii) Sketch the graph of $y = \left|f(x)\right|$, clearly showing all important features. **2**

(iv) State whether $\left|f(x)\right|$ has an inverse function. Justify your answer. **1**

End of Question 11

Question 12 (16 marks) Start on a NEW page **Marks**

(a) Find $\displaystyle\int_0^{\frac{\pi}{2}} \sin 5x \cos 3x \, dx$. **3**

(b) By making the substitution $t = \tan \dfrac{\theta}{2}$, or otherwise, show that **2**

$$\operatorname{cosec}\theta + \cot\theta = \cot\frac{\theta}{2}.$$

(c) The roots of the polynomial $P(x) = x^3 + qx^2 + rx + t$ are in an arithmetic progression. **3**

By letting the roots be $\alpha - d$, α and $\alpha + d$, or otherwise, show that

$$2q^3 - 9qr + 27t = 0.$$

(d) Determine the number of ways that ten people can be arranged in a circle such that no two of three particular people, Alice, Bridgette and Chloe, are to sit together. **2**

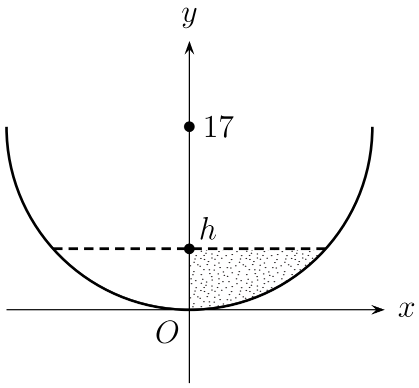
Examination continues overleaf...

(e) The left-hand diagram shows the lower half of the circle

$$x^2 + (y - 17)^2 = 289.$$

The shaded area in this diagram is bounded by the semicircle, the line $y = h$, and the y-axis.

The right-hand diagram shows a hemispherical bowl of radius 17 cm containing water of depth h cm.



(i) Show that the volume V formed when the shaded area in the left-hand diagram is rotated around the y-axis is given by **3**

$$V = 17\pi h^2 - \frac{\pi h^3}{3}.$$

(ii) The bowl is filled with water at a constant rate of 5 $\text{cm}^3\,\text{s}^{-1}$. **3**

Find the exact rate at which the water level is rising when the water level is 7 cm.

End of Question 12

Question 13 (15 marks) Start on a NEW page **Marks**

(a) Use mathematical induction to prove that **3**

$$\frac{3}{2^2 \times 1^2} + \frac{5}{3^2 \times 2^2} + \cdots + \frac{2n-1}{n^2(n-1)^2} = 1 - \frac{1}{n^2}$$

for all integers $n \geq 2$.

(b) A plane needs to travel to a destination that is on a bearing of 072°. The engine is set to fly at a constant 188 km/h. However, there is a wind from northwest with a constant speed of 49 km/h. **3**

On what constant bearing, to the nearest degree, should the direction of the plane be set in order to reach the destination?

(c) (i) For positive integers n and k, where $k \leq n$, show *algebraically* that **2**

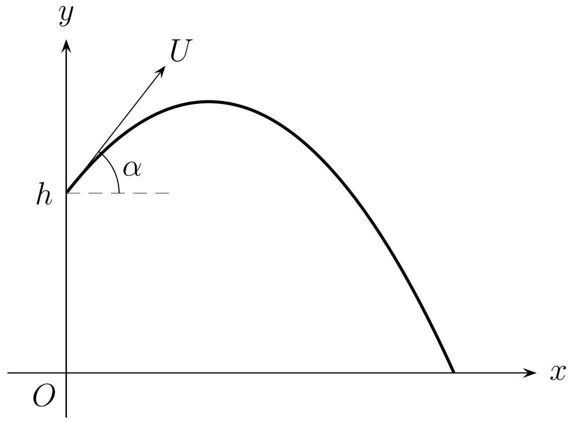
$$\frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1},$$

where $\binom{n}{k} = {}^nC_k$.

(ii) Prove the above identity by providing a *combinatorial* argument (i.e. by providing a *counting* argument). **2**

Examination continues overleaf...

(d) A military combat aircraft is flying with speed U m/s in a direction inclined at an angle α above the horizontal. When the aircraft is at height h m, a bomb is dropped. The diagram below shows the trajectory of the bomb. **3**



The bomb's displacement vector, t seconds after it is dropped, is

$$\underset{\sim}{r}(t) = \begin{pmatrix} Ut\cos\alpha \\ -\dfrac{gt^2}{2} + Ut\sin\alpha + h \end{pmatrix}.$$

(Do NOT prove this.)

Show that

$$gR = \frac{1}{2}U^2\sin 2\alpha + U\cos\alpha\sqrt{2gh + U^2\sin^2\alpha},$$

where R is the horizontal range of the bomb.

(e) The function $g(x) = x^3 + 2x + 3$ passes through the point $(1, 6)$. **2**

Find the exact gradient of the tangent to $f(x) = e^{g^{-1}(x)}$ at the point where $x = 6$.

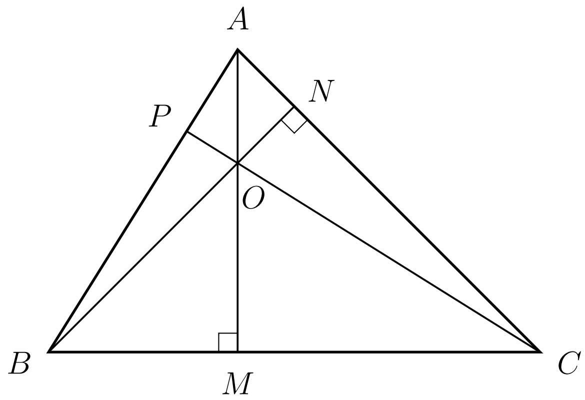
End of Question 13

Question 14 (14 marks) Start on a NEW page **Marks**

(a) In $\triangle ABC$, AM is perpendicular to BC and BN is perpendicular to AC. **3**

Let O be the point of intersection of AM and BN.

The line from C passing through O meets AB at P.



Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.

By using vector methods, show that CP is perpendicular to AB.

Examination continues overleaf...

(b) Two chemicals, $\mathcal{C}_1$ and $\mathcal{C}_2$, are put together in a solution, where they react to form a solid, $\mathcal{S}$. The rate of increase of the mass, m kg, of $\mathcal{S}$ is *proportional* to the product of the masses of *unreacted* $\mathcal{C}_1$ and $\mathcal{C}_2$ present at time t minutes.

It takes 1 kg of $\mathcal{C}_1$ and 3 kg of $\mathcal{C}_2$ to form 4 kg of $\mathcal{S}$.

Initially, 2 kg of $\mathcal{C}_1$ and 3 kg of $\mathcal{C}_2$ are put together in solution, and 1 kg of $\mathcal{S}$ forms in 1 minute.

(i) Carefully prove that **3**

$$\frac{dm}{dt} = k(8 - m)(4 - m),$$

where k is a constant.

(ii) Given that **3**

$$\frac{1}{(8 - x)(4 - x)} = -\frac{1}{4(8 - x)} + \frac{1}{4(4 - x)},$$

show that

$$t = \frac{1}{\ln \frac{7}{6}} \ln\left(\frac{8 - m}{8 - 2m}\right), \text{ for } 0 \le m < 4.$$

(iii) Find the exact mass of $\mathcal{S}$ formed after 2 minutes. **2**

(c) Consider three points $A(1, 1)$, $B(2, 2)$ and $C(3, y)$, where $y \ge 3$. **3**

Find the y-coordinate of C such that $\angle ACB$ is maximised.

End of paper

2024 Year 12 Mathematics Extension 1 Trial Marking Guidelines

Section I

Multiple-choice Answer Key

Question	Answer
1	D
2	A
3	A
4	A
5	C
6	B
7	C
8	C
9	D
10	B

Question 1 (1 mark)

$$\sqrt{2^2+3^2}=\sqrt{13} \quad \therefore \text{(D)} \quad ✓$$

Question 2 (1 mark)

$$\begin{aligned}\int_0^{\frac{3}{2}} \frac{1}{\sqrt{9-x^2}}\,dx &= \left[\sin^{-1}\left(\frac{x}{3}\right)\right]_0^{\frac{3}{2}} \\ &= \sin^{-1}\left(\frac{1}{2}\right)-\sin^{-1}(0) \\ &= \frac{\pi}{6} \quad \therefore \text{(A)} \quad ✓\end{aligned}$$

Question 3 (1 mark)

$\sin x - \sqrt{3}\cos x = R\sin(x-\alpha)$, where

$$R=\sqrt{1^2+(\sqrt{3})^2}=2$$

$$\tan\alpha = \frac{\sqrt{3}}{1} \quad \therefore \alpha=\frac{\pi}{3} \quad (0<\alpha<\frac{\pi}{2})$$

$\therefore$ (A) ✓

Note: If a question *explicitly* asks you to express a given expression in the form

$$R\sin(x\pm\alpha) \text{ or } R\cos(x\pm\alpha),$$

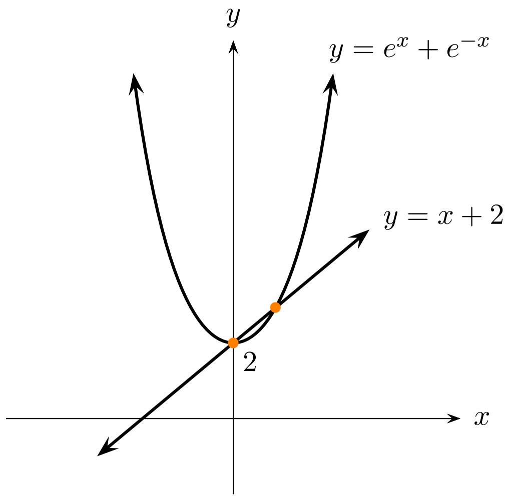
then full working must be shown; i.e. the above is *not* sufficient.
See 2022 HSC Mathematics X1 Q11(e) and the NESA Solutions.

Question 4 (1 mark)

$$x=1+t \implies t=x-1$$

$$\begin{aligned}y &= \frac{1-t}{1+t} \\ &= \frac{1-(x-1)}{1+x-1} \\ &= \frac{2-x}{x} \\ &= \frac{2}{x}-1 \quad \therefore \text{(A)} \quad ✓\end{aligned}$$

Question 5 (1 mark)



From the graph, $e^x + e^{-x} = x + 2$ has 2 solutions

$\therefore$ (C) ✓

Note: $y = e^x + e^{-x}$ can be sketched by considering $y = f(x) + g(x)$.

Question 6 (1 mark)

$$\begin{aligned}\int (2\cos^2 3x - 1)^2\,dx &= \int \cos^2 6x\,dx \\ &= \int \frac{1}{2}(\cos 12x + 1)\,dx \\ &= \frac{1}{2}\left(\frac{1}{12}\sin 12x + x\right) + C \\ &= \frac{1}{24}\sin 12x + \frac{1}{2}x + C \quad \therefore \text{(B)} ✓\end{aligned}$$

Question 7 (1 mark)

At the point $(0, -2)$, $\dfrac{dy}{dx} > 1$ (since the tangent there is steeper than $y = x$)

Substitute $(0, -2)$

(A): $\dfrac{dy}{dx} = 1$ ✗ (C): $\dfrac{dy}{dx} = 5$

(B): $\dfrac{dy}{dx} = 1$ ✗ (D): $\dfrac{dy}{dx} = 1$ ✗

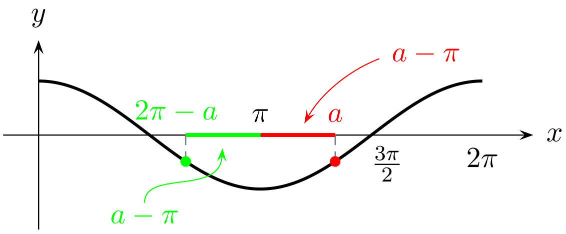
$\therefore$ (C) ✓

Note:
To check, let $\dfrac{dy}{dx} = 1$ in (C).

$$y^2 - xy + 1 = 1 \implies y(y - x) = 0 \quad \therefore y = 0,\ y = x$$

$\therefore$ Along the lines $y = 0$ and $y = x$ (and nowhere else), the slope of the tangents is 1.

Question 8 (1 mark)



In the diagram, the distance between π and a is $a-\pi$.

By symmetry, the green distance is $a-\pi$. The x-coordinate at the left side of the green bar is $\pi-(a-\pi)=2\pi-a$.

$\therefore \cos a = \cos(2\pi - a)$ (same y-value)

$$\therefore \cos^{-1}(\cos a) = \cos^{-1}(\cos(2\pi - a))$$
$$= 2\pi - a \quad \therefore \text{(C)}$$

Note: $f^{-1}(f(x)) = x$ for all x in the domain of $f(x)$ for which $f^{-1}(x)$ exists.

In this case, $\cos^{-1}(\cos x) = x$ for all $x \in [0,\pi]$, and observe that $a \notin [0,\pi]$ but $2\pi - a \in [0,\pi]$, so $\cos^{-1}(\cos a) \neq a$ and $\cos^{-1}(\cos(2\pi - a)) = 2\pi - a$.

Question 9 (1 mark)

$$\frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{\left(\text{proj}_{\underset{\sim}{v}}\underset{\sim}{u}\right)\cdot\left(\text{proj}_{\underset{\sim}{u}}\underset{\sim}{v}\right)} = \frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{\left(\dfrac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{|\underset{\sim}{v}|^2}\underset{\sim}{v}\right)\cdot\left(\dfrac{\underset{\sim}{v}\cdot\underset{\sim}{u}}{|\underset{\sim}{u}|^2}\underset{\sim}{u}\right)}$$

$$= \frac{\cancel{\underset{\sim}{u}\cdot\underset{\sim}{v}}}{\dfrac{(\underset{\sim}{u}\cdot\underset{\sim}{v})^2}{|\underset{\sim}{v}|^2\,|\underset{\sim}{u}|^2}\cancel{(\underset{\sim}{v}\cdot\underset{\sim}{u})}}$$

$$= \frac{|\underset{\sim}{v}|^2\,|\underset{\sim}{u}|^2}{(\underset{\sim}{u}\cdot\underset{\sim}{v})^2}$$

$$= \frac{|\underset{\sim}{v}|^2\,|\underset{\sim}{u}|^2}{|\underset{\sim}{u}|^2\,|\underset{\sim}{v}|^2\cos^2\theta}$$

$$= \frac{1}{\cos^2\theta}$$

$$= \sec^2\theta \quad \therefore \text{(D)}$$

Question 10 (1 mark)

Given $B : R : G = 1 : 4 : 2$ and $G : Y : O = 1 : 3 : 6 = 2 : 6 : 12$,

$\therefore B : R : G : Y : O = 1 : 4 : 2 : 6 : 12$

$\therefore$ There are $1 + 4 + 2 + 6 + 12 = 25$ parts,
with each part containing $400 \div 25 = 16$ balls.

$\therefore$ There are 16 blue, 64 red, 32 green, 96 yellow, and 192 orange balls.

To guarantee that at least 50 balls of one colour are drawn, consider the worst case scenario; i.e. 16 blue, 49 red, 32 green, 49 yellow, and 49 orange balls are drawn.

Drawing 1 more ball would ensure the criterion is met.

$\therefore 16 + 49 + 32 + 49 + 49 + 1 = 196$ balls must be drawn. $\therefore$ **(B)** ✓

Question 11 (a)

Comments
Done pretty well. Some students simplified $(\sin^{-1} x)^2$ as $\sin^{-2} x$, which is incorrect.

Criteria	Marks
• Provides correct solution using chain rule.	2
• Correctly differentiates u^3 or $\sin^{-1} x$.	1

Sample answer:

$$\frac{d}{dx}(\sin^{-1} x)^3$$

$$= 3(\sin^{-1} x)^2 \cdot \frac{\mathrm{d}}{\mathrm{dx}} \sin^{-1} x \ ✓$$

$$= \frac{3(\sin^{-1} x)^2}{\sqrt{1 - x^2}} \ ✓$$

Question 11 (b)

Comments
• When solving inequalities, both sides must be multiplied by a positive number so that the inequality sign stays the same. • A correct diagram (i.e. with correct concavity of the parabola) must be drawn to solve the quadratic inequation. • The denominator of a fraction cannot be zero.

Criteria	Marks
• Provides correct solution, considering $x \neq 2$.	3
• Correctly solves the quadratic inequality.	2
• Multiplies both sides by $(x-2)^2$.	1

Sample answer:

$$\frac{3}{x-2} \leq 1$$

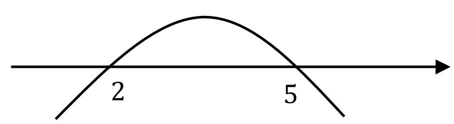
$$3(x-2) \leq (x-2)^2 \quad ✓$$

$$3(x-2) - (x-2)^2 \leq 0$$

$$(x-2)[3-(x-2)] \leq 0$$

$$(x-2)(5-x) \leq 0$$

$$x \leq 2 \quad \text{or} \quad x \geq 5 \quad ✓$$



$$\because x-2 \neq 0, x \neq 2$$

$$\therefore x < 2 \quad \text{or} \quad x \geq 5 \quad ✓$$

Note that:

The leading coefficient of $(x-2)(5-x)$ is negative, so the graph is concave down.

Question 11 (c)

Comments
Done generally well. • Some students forgot to multiply by $\frac{1}{2}$ or $\frac{1}{3}$. • Some students did not substitute $u = e^x$ at the end, so 1 mark was deducted.

Criteria	**Marks**
• Provides correct solution in terms of x.	3
• Attempts to integrate to have an expression involving $\arctan \square$.	2
• Rewrites the integrand using substitutions.	1

Sample answer:

$u = e^x$

$du = e^x dx$

$$\int \frac{du}{9 + 4u^2} \quad ✓$$

$$= \frac{1}{2} \int \frac{2}{3^2 + (2u)^2} du$$

$$= \frac{1}{2} \cdot \frac{1}{3} \arctan\left(\frac{2u}{3}\right) + C \quad ✓$$

$$= \frac{1}{6} \arctan\left(\frac{2e^x}{3}\right) + C \quad ✓$$

Question 11 (d)(i)

Comments
• Read the question carefully. $x > 2$ is the domain of $f(x)$ given in the question.

Criteria	Marks
• Provides correct solution.	1

Sample answer:

$y > 2$ ✓

Question 11 (d)(ii)

Comments: Done generally well.

Criteria	Marks
• Provides correct solution.	3
• Attempts to make y the subject.	2
• Swaps x and y.	1

Sample answer:

$f^{-1}(x)$:

$$x = \frac{3}{y-2} - 5$$ ✓

$$x + 5 = \frac{3}{y-2}$$

$$y - 2 = \frac{3}{x+5}$$ ✓

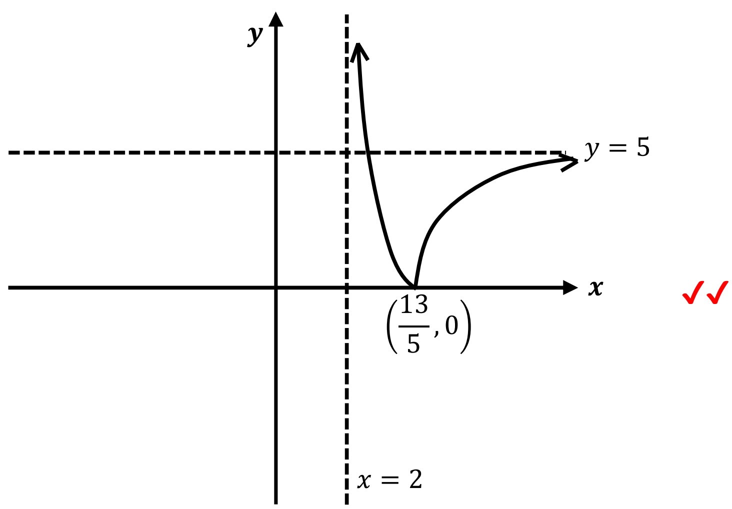
$$y = \frac{3}{x+5} + 2$$ ✓

Question 11 (d) (iii)

Comments:
• Read the question carefully. $x > 2$ is the domain of $f(x)$ given in the question.

Criteria	Marks
• Provides correct graph, showing all key features (i.e., asymptotes, intercepts). Carry on error from part (i) will be awarded.	2
• Provides correct graph but some key features are not shown.	1

Sample answer:



Question 7 (d) (iv)

Criteria	Marks
• Provides correct justification.	1

Sample answer:

No, because $y = |f(x)|$ is not a one-to-one function. ✓
(The given function $y = |f(x)|$ does not pass the horizontal line test.)

Feedback for Q12 Year 12 Ext 1 Trial 2024

(a) Mostly answered correctly if the first step was completed. Care needs to be taken with the negative values and for trig graph values. Marks allocated as indicated on the solutions.

(b) Mostly answered correctly. See solutions for the shorter way to complete using the reciprocals. Several students used a longer unnecessary solution. As this a "show" question, solution should include LHS = and RHS =

(c) Many students used a longer unnecessary solution. See solutions for the shorter way to complete - only needing to find α (using the sum of the roots), which is one of the roots of the polynomial and hence $P(\alpha)=0$.

(d) Several solutions (see solutions). Only a small number of students answered this correctly.

(e) (i) Mostly answered correctly if the first step was completed. Several students used a longer unnecessary solution. Hint: for this question, expand the binomial with power of 2 and simplify BEFORE integrating. Marks allocated as indicated on the solutions.

(ii) Marks allocated as indicated on the solutions. Some students missed taking the reciprocal of $\frac{dV}{dH}$. Note: $\frac{5}{189\pi} \neq \frac{5}{189}\pi$. Final answer is correctly written as $\frac{5}{189\pi}$ **not** $\frac{5}{189}\pi$, since $\frac{5}{189}\pi = \frac{5\pi}{189}$, however no marks were deducted if all other working was correct.

Q12 Ext. 1 Trial 2024

(a) $\int_0^{\frac{\pi}{2}} \sin 5x \cos 3x \, dx$

$= \int_0^{\frac{\pi}{2}} \frac{1}{2} (\sin 8x + \sin 2x) \, dx$ ✓1

$= \frac{1}{2} \left(\int \frac{8 \sin 8x}{8} dx + \int \frac{2 \sin 2x}{2} dx \right)$

$= \frac{1}{2} \left[-\frac{1}{8} \cos 8x - \frac{1}{2} \cos 2x \right]_0^{\frac{\pi}{2}}$ ✓1

$= \frac{1}{2} \left(\left(-\frac{1}{8} \cos 4\pi - \frac{1}{2} \cos \pi \right) - \left(-\frac{1}{8} - \frac{1}{2} \right) \right)$

$= \frac{1}{2} \left(-\frac{1}{8} + \frac{1}{2} + \frac{5}{8} \right)$

$= \frac{1}{2}$ ✓1

(b) $\operatorname{cosec} \theta + \cot \theta = \cot \frac{\theta}{2}$ Let $t = \tan \frac{\theta}{2}$

$\frac{1}{\sin \theta} + \frac{1}{\tan \theta} = \frac{1}{\tan \frac{\theta}{2}}$

$RHS = \frac{1}{t}$ ✓1

$LHS = \frac{1+t^2}{2t} + \frac{1-t^2}{2t}$

$= \frac{2}{2t}$ ✓1

$= \frac{1}{t}$

$= RHS$

Q12

(c) $p(x) = x^3 + qx^2 + rx + t$ AP

$a = 1 \quad b = q$

Sum of roots: $\alpha - d + \alpha + \alpha + d = 3\alpha$

$= -\frac{b}{a}$

$\therefore 3\alpha = -q$

$\alpha = \frac{-q}{3}$ /1

$p(\alpha) = 0 \quad \therefore p\left(\frac{-q}{3}\right) = 0$

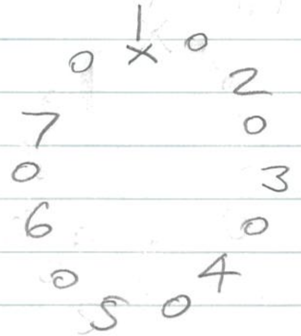
$\left(\frac{-q}{3}\right)^3 + q\left(\frac{-q}{3}\right)^2 + r\left(\frac{-q}{3}\right) + t = 0$ /1

$\frac{-q^3}{27} + \frac{q^3}{9} - \frac{rq}{3} + t = 0$

$-q^3 + 3q^3 - 9rq + 27t = 0$ /1

$2q^3 - 9qr + 27t = 0$

(d)



Seat 7 and place A, B, C in gaps.
[Fix 1 of 7 others and arrange 6 remaining of 7 others,
then 7 gaps → 7 ways to place Alice
6 gaps → 6 ways to place Bridgette
5 gaps → 5 way to place Chloe]

No. of ways $= 6! \times 7 \times 6 \times 5$

$= 151\,200$ /2

OR

No. of ways $= 7! \times {}^6C_2 \times 2!$

$= 151\,200$

OR

No. of ways $= 6! \times {}^7C_3 \times 3!$

$= 151\,200$

Several solutions

Q12

(e)(i) $V = \pi \int_0^h [f(y)]^2 \, dy$

$= \pi \int x^2 \, dy$

$= \pi \int (289 - (y-17)^2) \, dy$ ✓

$= \pi \int (289 - (y^2 - 34y + 289)) \, dy$

$= \pi \int (-y^2 + 34y) \, dy$ ✓

$= \pi \left[\frac{-y^3}{3} + 17y^2 \right]_0^h$

$= \pi \left(-\frac{h^3}{3} + 17h^2 - (0+0) \right)$ ✓

$\therefore V = 17\pi h^2 - \frac{\pi h^3}{3}$

(ii) $\frac{dV}{dt} = 5 \text{cm}^3/\text{s}$ $\qquad \frac{dh}{dt} = ?$

$\frac{dV}{dh} = 34\pi h - \pi h^2$ ✓

when $h = 7$

$\frac{dV}{dh} = 34\pi(7) - \pi(49)$

$= 238\pi - 49\pi$

$= 189\pi$

$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$ ✓

$= 5 \times \frac{1}{189\pi}$

$= \frac{5}{189\pi}$ cm/s ✓

Yr12 Ext 1 Trial 2024

Q13

a) Prove true for $n=2$

Many students had problems with the signs ∴ didn't get full mark.

$$\text{LHS} = \frac{2(2)-1}{2^2(2-1)^2} = \frac{3}{4}$$

$$\text{RHS} = 1 - \frac{1}{4} = \frac{3}{4}$$

LHS = RHS

∴ true for $n=2$

Assume true for $n=k$

$$\frac{3}{2^2\times1^2} + \frac{5}{3^2\times2^2} + \cdots + \frac{2k-1}{k^2(k-1)^2} = 1 - \frac{1}{k^2}$$

Prove true for $n=k+1$

$$\frac{3}{2^2\times1^2} + \frac{5}{3^2\times2^2} + \cdots \frac{2k-1}{k^2(k-1)^2} + \frac{2(k+1)-1}{(k+1)^2(k+1-1)^2} = 1 - \frac{1}{(1+k)^2}$$

$$\text{LHS} = 1 - \frac{1}{k^2} + \frac{2k+2-1}{(k+1)^2(k)^2}$$

$$= 1 - \frac{1}{k^2} + \frac{2k+1}{(k+1)^2(k)^2}$$

$$= 1 + \frac{2k+1}{(k+1)^2k^2} - \frac{1(k+1)^2}{(k+1)^2(k^2)}$$

$$= 1 + \frac{2k+1-1(k^2+2k+1)}{k^2(k+1)^2}$$

$$= 1 + \frac{2k+1-k^2-2k-1}{k^2(k+1)^2}$$

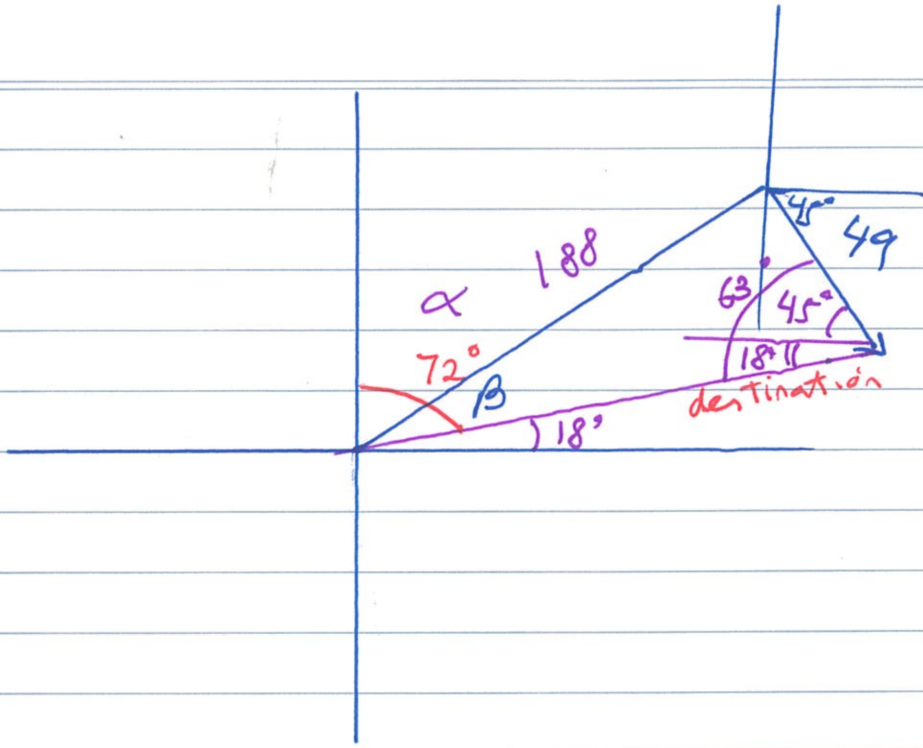
$$= 1 - \frac{k^2}{k^2(k+1)^2}$$

$$= 1 - \frac{1}{(k+1)^2}$$

= RHS

By M.I true for $n \geq 2$

Q13b)



$$\frac{\sin \beta}{49} = \frac{\sin 63°}{188}$$

$$\beta = 13.42°$$

$$\alpha = 72 - 13.42°...$$
$$= 59°$$

Bearing 059°T

students found this question very challenging. They couldn't draw the picture correctly.

c) i) LHS $= \frac{k}{n}\left(\frac{n!}{(n-k)!\,k!}\right)$

$$= \frac{k\,n\,(n-1)!}{n\,(n-k)!\,k\,(k-1)!}$$

$$= \frac{(n-1)!}{(n-k)!\,(k-1)!}$$

or $\frac{(n-1)!}{(n-k-1+1)!\,(k-1)!}$

$= \frac{(n-1)!}{(n-1-(k-1))!\,(k-1)!}$

= RHS

RHS $= \frac{(n-1)!}{(n-1-(k-1))!\,(k-1)!}$

$$= \frac{(n-1)!}{(n-1-k+1)!\,(k-1)!}$$

$$= \frac{(n-1)!}{(n-k)!\,(k-1)!}$$

To get full marks you had to show how LHS = RHS you can't just state it without any steps

ii) $k\binom{n}{k} = n\binom{n-1}{k-1}$

- from n people, select k people and make one of them a captain

$${}^{n}C_{k} \times {}^{k}C_{1} = \binom{n}{k}k$$

or

first choose a Leader from n people, then from remaining (n-1) people choose (k-1) to get a group of k people all together.

$${}^{n}C_{1} \times {}^{(n-1)}C_{k-1} = n\binom{n-1}{k-1}$$

$$\therefore\ k\binom{n}{k} = n\binom{n-1}{k-1}$$

$$\frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$$

Students found this question very challenging as well. There was either no attempt or wrong working.

13d) $x = ut\cos\alpha$

$$y = -\frac{gt^2}{2} + ut\sin\alpha + h$$

$y = 0$

$$-gt^2 + 2ut\sin\alpha + 2h = 0$$

$$t = \frac{-2u\sin\alpha \pm \sqrt{4u^2\sin^2\alpha - 4(-g)(2h}}{-2g}$$

$$= \frac{-2u\sin\alpha \pm \sqrt{4u^2\sin^2\alpha + 8gh}}{-2g}$$

$$= \frac{-2u\sin\alpha \pm 2\sqrt{u^2\sin^2\alpha + 2gh}}{-2g}$$

$$= \frac{u\sin\alpha \pm \sqrt{u^2\sin^2\alpha + 2gh}}{g}$$

* Some students were careless with ± signs

$t > 0$ ∴

$$= \frac{u\sin\alpha + \sqrt{u^2\sin^2\alpha + 2gh}}{g}$$

$$x = u\cos\alpha\left(\frac{u\sin\alpha + \sqrt{u^2\sin^2\alpha + 2gh}}{g}\right)$$

$x = R$ $\quad Rg = u^2\cos\alpha\sin\alpha + u\cos\alpha\sqrt{u^2\sin^2\alpha + 2gh}$

$$= \frac{1}{2}u^2\sin 2\alpha + u\cos\alpha\sqrt{u^2\sin^2\alpha + 2gh}$$

e) This questions was done poorly as well.

$$f'(x) = \left(g^{-1}(x)\right)' e^{g^{-1}(x)}$$

$$g'(x) = 3x^2 + 2$$

$$\frac{dg}{dx} = 3x^2 + 2$$

$$\frac{dx}{dg} = \frac{1}{3x^2+2}$$

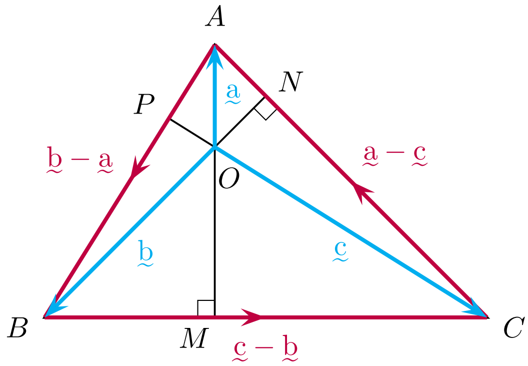
at $x = 1 \rightarrow \frac{dx}{dg} = \frac{1}{5}$

at $x = 6$ $\quad g^{-1}(x) = 1$

$\therefore$

$$f'(x) = \frac{1}{5} e^{1}$$

Question 14 (a) (3 marks)



$\overrightarrow{CP} = k\underset{\sim}{c}$, where k is a scalar

$\overrightarrow{CP} \cdot \overrightarrow{AB} = k\underset{\sim}{c} \cdot (\underset{\sim}{b} - \underset{\sim}{a})$

Given $\underset{\sim}{b} \cdot (\underset{\sim}{a} - \underset{\sim}{c}) = 0$ ✓ and $\underset{\sim}{a} \cdot (\underset{\sim}{c} - \underset{\sim}{b}) = 0$ ✓ (since $AM \perp BC$ and $BN \perp AC$)

$\therefore \underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{b} \cdot \underset{\sim}{c}$ (1) and $\underset{\sim}{a} \cdot \underset{\sim}{c} = \underset{\sim}{a} \cdot \underset{\sim}{b}$ (2)

Sub (1) → (2) :

$\underset{\sim}{a} \cdot \underset{\sim}{c} = \underset{\sim}{b} \cdot \underset{\sim}{c}$

$\therefore \underset{\sim}{c} \cdot (\underset{\sim}{a} - \underset{\sim}{b}) = 0$

$$\begin{aligned}\therefore \overrightarrow{CP} \cdot \overrightarrow{AB} &= -k\underset{\sim}{c} \cdot (\underset{\sim}{a} - \underset{\sim}{b}) \\ &= -k \times 0 \\ &= 0\end{aligned}$$

$\therefore CP \perp AB$ ✓

Marking Scheme
✓ [1] for writing $\underset{\sim}{b} \cdot (\underset{\sim}{a} - \underset{\sim}{c}) = 0$, or equivalent merit
✓ [1] for writing $\underset{\sim}{a} \cdot (\underset{\sim}{c} - \underset{\sim}{b}) = 0$, or equivalent merit
✓ [1] for correctly showing the required result

Marker's Comments:

- Mixed responses. Responses which used the facts that $AM \perp BC$ and $BN \perp AC$ to deduce that $\underset{\sim}{b} \cdot (\underset{\sim}{a} - \underset{\sim}{c}) = 0$ and $\underset{\sim}{a} \cdot (\underset{\sim}{c} - \underset{\sim}{b}) = 0$ were generally successful.
- Unsuccessful responses often resorted to overcomplicated methods.

Question 14 (b)(i) (3 marks)

Given $\mathcal{C}_1$ is $\frac{1}{4}$ of $\mathcal{S}$, $\therefore$ mass of $\mathcal{C}_1$ is $\frac{1}{4}m$ at time t minutes ✓

Similarly, mass of $\mathcal{C}_2$ is $\frac{3}{4}m$ at time t minutes ✓

$$\begin{aligned}\therefore \frac{dm}{dt} &= A\left(2-\frac{1}{4}m\right)\left(3-\frac{3}{4}m\right)\\ &= A\times\frac{1}{4}(8-m)\times\frac{3}{4}(4-m)\\ &= k(8-m)(4-m) \quad ✓\end{aligned}$$

Marking Scheme
✓ [1] for correctly showing that the mass of $\mathcal{C}_1$ at time t is $\frac{1}{4}m$
✓ [1] for correctly showing that the mass of $\mathcal{C}_2$ at time t is $\frac{3}{4}m$
✓ [1] for correctly showing the required result

Marker's Comments:

- Most students found this question challenging.
- Students are advised to read and learn the techniques shown in the above solution, and to reattempt this question until mastery.

Question 14 (b)(ii) (3 marks)

$$\frac{dm}{(8-m)(4-m)} = k\,dt$$

$$\int_0^m \frac{dm}{(8-m)(4-m)} = k\int_0^t dt$$

$$\int_0^m -\frac{1}{4(8-m)} + \frac{1}{4(4-m)}\,dm = kt$$

$$\frac{1}{4}\Big[\ln(8-m) - \ln(4-m)\Big]_0^m = kt \quad ✓$$

$$\frac{1}{4}\left[\ln\left(\frac{8-m}{4-m}\right)\right]_0^m = kt$$

$$\frac{1}{4}\left(\ln\left(\frac{8-m}{4-m}\right) - \ln 2\right) = kt$$

$$\frac{1}{4}\ln\left(\frac{8-m}{8-2m}\right) = kt \quad ✓$$

When $t = 1$, $m = 1$ $\quad \therefore k = \frac{1}{4}\ln\frac{7}{6}$

$$\therefore t = \frac{1}{4k}\ln\left(\frac{8-m}{8-2m}\right)$$

$$= \frac{1}{\ln\frac{7}{6}}\ln\left(\frac{8-m}{8-2m}\right) \quad ✓$$

Marking Scheme

✓	[1]	for correct separation of variables and integrating correctly
✓	[1]	for obtaining $\frac{1}{4}\ln\left(\frac{8-m}{8-2m}\right) = kt$
✓	[1]	for correctly showing the required result

Marker's Comments:

- Generally well done, although there were some careless errors made when integrating $-\frac{1}{4(8-m)} + \frac{1}{4(8-m)}$.
- Some students who used indefinite integrals did not write the constant of integration, or did not evaluate it using the initial conditions.
- Some students also did not use the given identity. It is important to see what results/facts are given in the question, and to be able to use them to make progress in the question.

Question 14 (b)(iii) (2 marks)

Let $t = 2$

$2\ln\frac{7}{6} = \ln\frac{8-m}{8-2m}$

$\ln\left(\frac{7}{6}\right)^2 = \ln\frac{8-m}{8-2m}$

$\frac{49}{36} = \frac{8-m}{8-2m}$ ✓

$392 - 98m = 288 - 36m$

$104 = 62m$

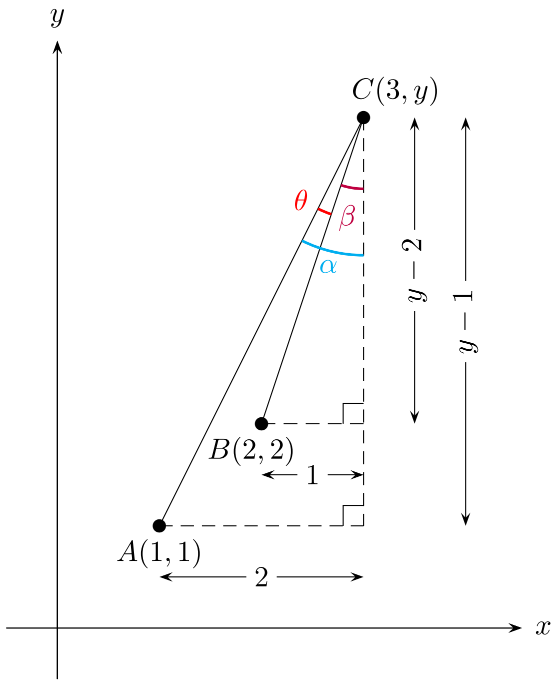
$\therefore m = \frac{52}{31}$ kg ✓

Marking Scheme
✓ [1] for substantial progress
✓ [1] for correct value of m

Marker's Comments:

- Generally well done.
- Some careless errors with index and log laws were made by a number of students.

Question 14 (c) (3 marks)



$$\tan\alpha = \frac{2}{y-1} \quad \text{and} \quad \tan\beta = \frac{1}{y-2}$$

$$\begin{aligned}\theta &= \alpha - \beta \\ &= \tan^{-1}\left(\frac{2}{y-1}\right) - \tan^{-1}\left(\frac{1}{y-2}\right) \quad ✓\end{aligned}$$

$$\begin{aligned}\frac{d\theta}{dy} &= \frac{-\frac{2}{(y-1)^2}}{1+\frac{4}{(y-1)^2}} - \frac{-\frac{1}{(y-2)^2}}{1+\frac{1}{(y-2)^2}} \\ &= \frac{1}{(y-2)^2+1} - \frac{2}{(y-1)^2+4}\end{aligned}$$

$$\text{Let } \frac{d\theta}{dy} = 0 \quad \therefore \frac{1}{(y-2)^2+1} = \frac{2}{(y-1)^2+4}$$

$$\begin{aligned}(y-1)^2+4 &= 2(y-2)^2+2 \\ y^2-2y+5 &= 2y^2-8y+10 \\ 0 &= y^2-6y+5 \\ 0 &= (y-5)(y-1) \\ \therefore y &= 5 \quad (\text{since } y \geq 3) \quad ✓\end{aligned}$$

y	4	5	6
$\frac{d\theta}{dy}$	$\frac{3}{65}$	0	$-\frac{5}{493}$
	↗	⟶	↘

$\therefore \angle ACB$ is maximised when $y = 5$ ✓

Marking Scheme
✓ [1] for correct expression of $\angle ACB$, or equivalent merit
✓ [1] for substantial progress thereafter
✓ [1] for correctly proving that $y = 5$ maximises $\angle ACB$

Question 14 (c) (3 marks) continued...

Marker's Comments:

- Majority of the students found this question very challenging.
- Unsuccessful responses often resorted to overcomplicated methods.
- Students are advised to read and learn the techniques shown in the above solution, and to reattempt this question until mastery.