Variations on a Theme of Toloza

by Derek Buchanan

In 2023 a generalisation of Toloza's formula for π was discovered. Toloza's formula for π is $2+\binom{2}{2}^{-1}+\binom{3}{2}^{-1}-\binom{4}{2}^{-1}-\binom{5}{2}^{-1}+\binom{6}{2}^{-1}+\binom{7}{2}^{-1}-\cdots=\pi$ and relates π to one diagonal of Pascal's triangle [1]. I gave a simplified proof in [2].

It can also be written as $2+\sum_{n=1}^{\infty}\sum_{r=0}^{1}(-1)^{n-1}\binom{2n+r}{2}^{-1}=\pi$.

Suppose we define $T_p := \frac{2p}{2p-1}+\sum_{n=1}^{\infty}\sum_{r=0}^{2p-1}(-1)^{n-1}\binom{2pn+r}{2p}^{-1}$ for positive integers p.

Toloza's formula is now $T_1=\pi$.

One may explore a generalisation of the formula to find similar formulae from the

Proposition. For $p>1,\ T_p=2\pi\sum_{i=1}^{p-1}(-1)^{i-1}\binom{2p-2}{i-1}(\csc\frac{\pi i}{2p}-1)$.

This has now been proved in [3].

What this means is that not only is π related to one diagonal of Pascal's triangle, but to infinitely many of them - in particular, every second diagonal.

For example, with $p=2$ we find that $T_2=2\pi(\binom{2}{0}\csc\frac{\pi}{4}-1)=2(\sqrt{2}-1)\pi$.

So instead of a 2-fold alternating sum as in Toloza's formula (where pairs of terms are added, then subtracted, then added, etc.,) we now have a 4-fold alternating sum

$$\frac{4}{3}+\binom{4}{4}^{-1}+\binom{5}{4}^{-1}+\binom{6}{4}^{-1}+\binom{7}{4}^{-1}-\binom{8}{4}^{-1}-\binom{9}{4}^{-1}-\binom{10}{4}^{-1}-\binom{11}{4}^{-1}+\cdots=2(\sqrt{2}-1)\pi$$

Likewise,

$$\begin{aligned}T_3&=2(\binom{4}{0}(\csc\frac{\pi}{6}-1)-\binom{4}{1}(\csc\frac{2\pi}{6}-1))\pi\\&=2(3+\csc\frac{\pi}{6}-4\csc\frac{2\pi}{6})\pi\\&=2(5-\frac{8}{3}\sqrt{3})\pi \text{ giving the 6-fold alternating sum}\end{aligned}$$

$$\frac{6}{5}+\binom{6}{6}^{-1}+\binom{7}{6}^{-1}+\binom{8}{6}^{-1}+\binom{9}{6}^{-1}+\binom{10}{6}^{-1}+\binom{11}{6}^{-1}-\binom{12}{6}^{-1}-\binom{13}{6}^{-1}-\binom{14}{6}^{-1}-\binom{15}{6}^{-1}-\binom{16}{6}^{-1}-\binom{17}{6}^{-1}+\cdots=2(5-\frac{8}{3}\sqrt{3})\pi$$

and so on.

References

[1] https://docs.google.com/file/d/1mnJBpUsWjLDhmpT2-6u9qDpGvtSHiWUTvbYCG2pEXtWGovrluS9u57E0paJf

[2] https://www.angelfire.com/ab7/fourunit/binomial-formula-for-pi.pdf

[3] https://4unitmaths.com/generalised-toloza-sum-proof.pdf