Vectors in the new Syllabus

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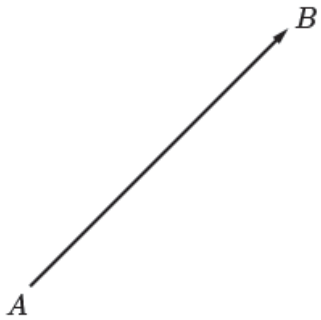
Quantities which have only magnitude are called scalars. Quantities which have magnitude and direction are called vectors.

$\overrightarrow{AB}$ is the position vector of B relative to A and is the vector which emanates from A and terminates at B and has length $|\overrightarrow{AB}|$:



Vectors which don't have to start from a particular point are called displacement vectors. They can be moved around, and remain the same so long as their magnitude or direction don't change.

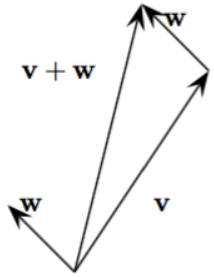
Two vectors are equal if they have the same magnitude and direction. They will be parallel and have the same length.

$\overrightarrow{BA} = -\overrightarrow{AB}$ has the same magnitude but opposite direction to $\overrightarrow{AB}$.

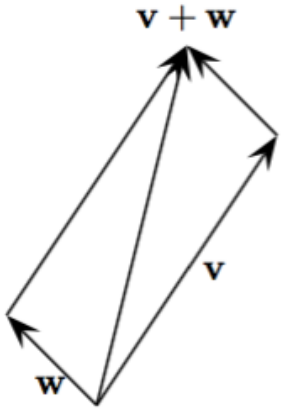
Other notations: If $O = (0, 0, 0), A = (a_1, a_2, a_3)$ then $\mathbf{a} = \overrightarrow{OA} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \underset{\sim}{a} = \vec{a}$

Operations with vectors

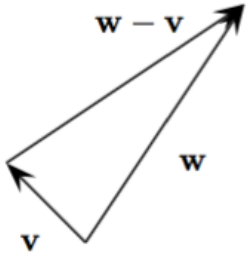
Addition (triangle law): To construct $\mathbf{v} + \mathbf{w}$ draw $\mathbf{w}$ at the arrowhead end of $\mathbf{v}$. Join beginning of $\mathbf{v}$ to arrowhead end of $\mathbf{w}$.



Parallelogram law Just construct the diagonal as the sum.



Vector subtraction:



We note that $\mathbf{v} + (\mathbf{w} - \mathbf{v}) = \mathbf{w}$

Zero vector: $\mathbf{a} - \mathbf{a} = \mathbf{0}$

Scalar multiplication:

$k\mathbf{a}$ stretches $\mathbf{a}$ by factor k where k is a scalar.

$k > 0 \Rightarrow k\mathbf{a}$ is in the same direction as $\mathbf{a}$.

$k < 0 \Rightarrow k\mathbf{a}$ is in the opposite direction as $\mathbf{a}$.

Examples

$\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$
$\begin{pmatrix} -2 \\ -4 \\ -6 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Length of a vector

This is really just a generalisation of Pythagoras' Theorem.

If $O = (0,0,0), P = (x,y,z)$ then $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$ and $\overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Likewise, if $A = (x_1, y_1, z_1), B = (x_2, y_2, z_2)$ then $|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ and $\overrightarrow{AB} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}$

Eg., $\left| \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$

Operations on vectors in component form

$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \Rightarrow \mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}, \mathbf{a} - \mathbf{b} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{pmatrix}$

$k =$ a scalar $\Rightarrow k\mathbf{a} = \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix}$.

$\overrightarrow{OA} = \mathbf{a}, \overrightarrow{OB} = \mathbf{b} \Rightarrow \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.

Parallelism

$\mathbf{a} \parallel \mathbf{b}$ if $\mathbf{a} = k\mathbf{b}$ for some scalar k.

A, B, C are collinear if $\overrightarrow{AB} = k\overrightarrow{BC}$ for some scalar k.

Examples

1. Show A, B, C are collinear if $A = (1, 2, 3)$, $B = (2, 4, 6)$, $C = (4, 8, 12)$

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} = \frac{1}{2}\overrightarrow{BC}$$

2. Show A, B, C are not collinear if $A = (1, 2, 3)$, $B = (2, 4, 6)$, $C = (4, 8, 14)$

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \text{ and } \overrightarrow{BC} = \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix}.$$

Suppose $\overrightarrow{AB} = k\overrightarrow{BC}$. Then from the first component, $1 = 2k$ and $k = \frac{1}{2}$. But then in the third component, $8k = 4 \neq 3$. Hence the points are not collinear.

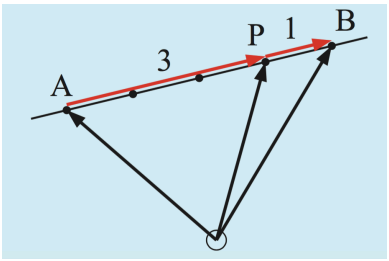
Division of a line segment

Suppose A, B, X lie on a straight line. Then X divides AB in ratio $a : b$ if $|\overrightarrow{AX}| : |\overrightarrow{XB}| = a : b$.

$(a > 0, b > 0) \Rightarrow$ internal division

$(a > 0, b < 0) \Rightarrow$ external division & $|\overrightarrow{AX}| : |\overrightarrow{XB}| = a : -b$.

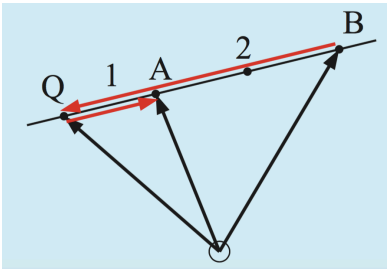
Example 1. $A = (-1, 4, 7)$, $B = (3, 0, 5)$. Find P if P divides AB in the ratio $3 : 1$.



$$\begin{aligned} \overrightarrow{OP} &= \overrightarrow{OA} + \overrightarrow{AP} \\ &= \overrightarrow{OA} + \tfrac{3}{4}\overrightarrow{AB} \\ &= \begin{pmatrix} -1 \\ 4 \\ 7 \end{pmatrix} + \tfrac{3}{4}\begin{pmatrix} 4 \\ -4 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 1 \\ \frac{11}{2} \end{pmatrix} \end{aligned}$$

Hence $P = (2, 1, \frac{11}{2})$

Example 2. $A = (-1, 4, 7)$, $B = (3, 0, 5)$. Find Q if Q divides BA externally in the ratio $3 : 1$.



$$\begin{aligned}\overrightarrow{OQ} &= \overrightarrow{OA} + \overrightarrow{AQ} \\ &= \overrightarrow{OA} + \tfrac{1}{2}\overrightarrow{BA} \\ &= \begin{pmatrix} -1 \\ 4 \\ 7 \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} -4 \\ 4 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ 6 \\ 8 \end{pmatrix}\end{aligned}$$

Hence $Q = (-3, 6, 8)$

Unit vectors

A unit vector is a vector of length 1. Base unit vectors are unit vectors which can be used as linear combinations to write any other vector. The most commonly used base unit vectors in 3-D are $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

The unit vector in direction of $\mathbf{u}$ is denoted $\hat{\mathbf{u}}$ and is equal to $\frac{\mathbf{u}}{|\mathbf{u}|}$.

Note too that a vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ can now be written as $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

We don't usually use the hat notation $\hat{\mathbf{u}}$ for $\mathbf{i}, \mathbf{j}, \mathbf{k}$ because they are already unit vectors by definition.

Scalar product

(Also called dot product)

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \Rightarrow \mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

Note the relationship to length: $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Eg., $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} = 1 \times 2 + 2 \times 4 + 3 \times 6 = 28$

$$\left|\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right| = \sqrt{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}} = \sqrt{1 \times 1 + 2 \times 2 + 3 \times 3} = \sqrt{14}$$

Angle between vectors: If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ then $\theta = \cos^{-1} \hat{\mathbf{a}} \cdot \hat{\mathbf{b}}$.

Proof. By the cosine rule, $|\mathbf{b} - \mathbf{a}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos\theta$. But $\mathbf{b} - \mathbf{a} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix}$

$\therefore (b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2$
$= a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - 2|\mathbf{a}||\mathbf{b}| \cos\theta$

$\therefore a_1 b_1 + a_2 b_2 + a_3 b_3 = \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta \therefore \cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

$\therefore \theta = \cos^{-1} \hat{\mathbf{a}} \cdot \hat{\mathbf{b}}$ □

Note that this can also be used to generalise the concept of angle to higher dimensions.

Perpendicular vectors

For nonzero vectors $\mathbf{a}, \mathbf{b}, \mathbf{a} \perp \mathbf{b} \Leftrightarrow \mathbf{a} \cdot \mathbf{b} = 0$ (since $\cos 90° = 0$)

Parallel vectors

For nonzero vectors $\mathbf{a}, \mathbf{b}, \mathbf{a} \parallel \mathbf{b} \Leftrightarrow \mathbf{a} \cdot \mathbf{b} = \pm|\mathbf{a}||\mathbf{b}|$ (since $\cos 0° = 1$ and $\cos 180° = -1$).

Examples.

1. Find the angle between $\mathbf{p} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$.

$\cos^{-1} \hat{\mathbf{p}} \cdot \hat{\mathbf{q}} = \cos^{-1} \frac{2 \times -1 + 3 \times 0 + -1 \times 2}{\sqrt{2^2+3^2+1^2}\sqrt{1^2+0^2+2^2}} = 119°$

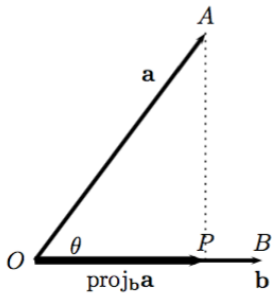
2. Find t such that $\begin{pmatrix} -1 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ t \end{pmatrix}$ are perpendicular.

$\begin{pmatrix} -1 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ t \end{pmatrix} = 0 \therefore -1 \times 2 + 5t = 0$ and so $t = \frac{2}{5}$.

3. Find t such that $\begin{pmatrix} -1 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ t \end{pmatrix}$ are parallel.

$\begin{pmatrix} -1 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ t \end{pmatrix} = -1 \times 2 + 5t = \pm\sqrt{1^2 + 5^2}\sqrt{2^2 + t^2}$
$\therefore (5t - 2)^2 = 26(4 + t^2) \therefore t = -10$

Projection vectors The vector projection of $\mathbf{a}$ in direction $\mathbf{b}$ is $(\mathbf{a}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}$



Proof.

$$\begin{aligned}\text{Proj}_{\mathbf{b}}\mathbf{a} &= |\mathbf{a}|\cos\theta\frac{\mathbf{b}}{|\mathbf{b}|} \text{ where } \theta \text{ is the angle between } \mathbf{a},\mathbf{b}\\ &= |\mathbf{a}|\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\frac{\mathbf{b}}{|\mathbf{b}|}\\ &= \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b} \text{ (actually this is probably the most useful form of this)}\\ &= (\mathbf{a}\cdot\frac{\mathbf{b}}{|\mathbf{b}|})\frac{\mathbf{b}}{|\mathbf{b}|}\\ &= (\mathbf{a}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}} \qquad \square\end{aligned}$$

$|\mathbf{a}|\cos\theta = \mathbf{a}\cdot\hat{\mathbf{b}}$ is called the scalar projection of $\mathbf{a}$ in direction $\mathbf{b}$.

The vector projection of $\mathbf{a}$ in a direction orthogonal to $\mathbf{b}$ is $\mathbf{a}-\text{Proj}_{\mathbf{b}}\mathbf{a}$.

Examples.

1. Find $\text{Proj}_{\mathbf{v}}\mathbf{u}$ where $\mathbf{u} = \begin{pmatrix}2\\-3\end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix}5\\6\end{pmatrix}$

$\text{Proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} = \frac{2\times5-3\times6}{5^2+6^2}\begin{pmatrix}5\\6\end{pmatrix} = -\frac{8}{61}\begin{pmatrix}5\\6\end{pmatrix}$

2. Find the scalar projection of $\mathbf{u}$ onto $\mathbf{v}$ from Example 1.

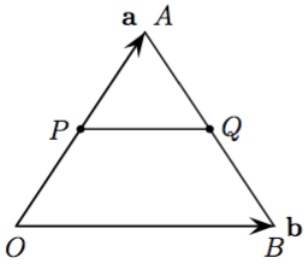
Rewriting the answer from Example 1 as a scalar multiple of a unit vector, $-\frac{8\sqrt{61}}{61}\begin{pmatrix}5/\sqrt{61}\\6/\sqrt{61}\end{pmatrix}$, this is just the number $-\frac{8\sqrt{61}}{61}$.

3. Find the vector projection of $\mathbf{u}$ in a direction orthogonal to $\mathbf{v}$ from Example 1.

$\mathbf{u} - \text{Proj}_{\mathbf{v}}\mathbf{u} = \begin{pmatrix}2\\-3\end{pmatrix} - -\frac{8}{61}\begin{pmatrix}5\\6\end{pmatrix} = \frac{27}{61}\begin{pmatrix}6\\-5\end{pmatrix}$

Geometric Proofs

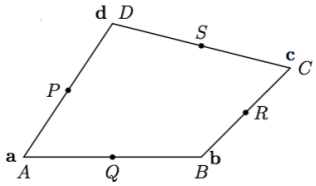
Theorem 1. The line joining the midpoint of two sides of a triangle is parallel to the third side and half its length.



Proof. $\overrightarrow{PQ} = \overrightarrow{PA} + \overrightarrow{AQ} = \frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{AB}) = \frac{1}{2}\overrightarrow{OB}\square$

If you thought that was quick, the next one is even quicker!

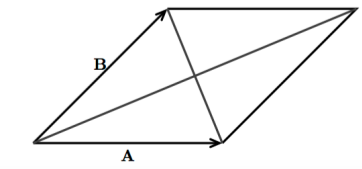
Theorem 2. If $ABCD$ is a convex quadrilateral then the midpoints form a parallelogram.



Proof. By Theorem 1, $\overrightarrow{PQ} = \frac{1}{2}\overrightarrow{DB} = \overrightarrow{SR}$. □

And that's it! (But think about WHY that's it.)

Theorem 3. The diagonals of a parallelogram meet at right angles if and only if it is a rhombus.

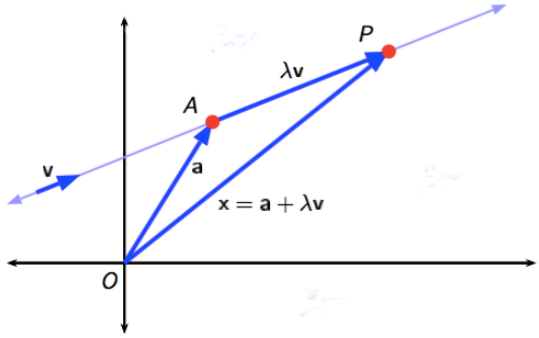


Proof. The diagonals are $\mathbf{A}+\mathbf{B}$ and $\mathbf{A}-\mathbf{B}$ and $(\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}-\mathbf{B}) = \mathbf{A}\cdot\mathbf{A} - \mathbf{B}\cdot\mathbf{B} = 0$
$\iff |\mathbf{A}| = \sqrt{\mathbf{A}\cdot\mathbf{A}} = \sqrt{\mathbf{B}\cdot\mathbf{B}} = |\mathbf{B}|$
And so the diagonals are perpendicular if and only if it is a rhombus. □

Theorem 4. The sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of the sides.

Proof. Using the same diagram as for Theorem 3, we have
$|\mathbf{A}+\mathbf{B}|^2+|\mathbf{A}-\mathbf{B}|^2$
$=(\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B})+(\mathbf{A}-\mathbf{B})\cdot(\mathbf{A}-\mathbf{B})$
$=\mathbf{A}\cdot\mathbf{A}+2\mathbf{A}\cdot\mathbf{B}+\mathbf{B}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{A}-2\mathbf{A}\cdot\mathbf{B}+\mathbf{B}\cdot\mathbf{B}$
$=2|\mathbf{A}|^2+2|\mathbf{B}|^2$ □

Vector parametric form of a line



Two lines $\mathbf{x_1}=\mathbf{a_1}+\lambda\mathbf{v_1}$ and $\mathbf{x_2}=\mathbf{a_2}+\lambda\mathbf{v_2}$ are respectively parallel or perpendicular if $\mathbf{v_1}$ and $\mathbf{v_2}$ are parallel or perpendicular.

Cartesian equations of a line

Suppose $\mathbf{x}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x_0\\y_0\\z_0\end{pmatrix}+\lambda\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$.

Then so long as v_1, v_2, v_3 are nonzero we may get the Cartesian equations for the line as

$\lambda=\frac{x-x_0}{v_1}=\frac{y-y_0}{v_2}=\frac{z-z_0}{v_3}$

Convex Combinations

To write interval AB as a convex combination of A and B,

$\mathbf{x}=\overrightarrow{OA}+\lambda\overrightarrow{AB}$ for $0\leq\lambda\leq 1$
$=\overrightarrow{OA}+\lambda(\overrightarrow{OB}-\overrightarrow{OA})$
$=\lambda\overrightarrow{OB}+(1-\lambda)\overrightarrow{OA}$

Example 1. Do $P(9,-2,-5)$ and $Q(-3,4,0)$ lie on the line $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$?

If $1 + 2\lambda = 9$ then $\lambda = 4$

Now check $2 - 4 = -2$ and $-1 - 4 = -5$. Hence P lies on the line.

If $1 + 2\lambda = -3$ then $\lambda = -2$.

Checking $2 - -2 = 4$ but $-1 - -2 = 1 \neq 0$ and hence Q does not lie on the line.

Example 2. Give a vector equation of a line parallel to the line from Example 1 passing through $(1, 2, 3)$.

Let it have the same direction vector $\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ and with $\lambda = 0$ the position vector will be $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

Hence the line is $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$

Example 3. Find a Cartesian equation of the line

$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$\lambda = \frac{x-1}{3} = \frac{y-2}{4}$ and so $4x - 4 = 3y - 6$

and so in general form $4x - 3y + 2 = 0$

Example 4. Do $\ell_1 : \mathbf{x}_1 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ and $\ell_2 : \mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ intersect?

Method 1. If $\mathbf{x}_1 = \mathbf{x}_2$ then $\mu \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} - \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

So solving simultaneously $2\mu - 2\lambda = 1$ and $-\mu - \lambda = -2$ we have that $\lambda = \frac{3}{4}$ and $\mu = \frac{5}{4}$.

But checking that $2\mu - \lambda = \frac{5}{2} - \frac{3}{4} = \frac{7}{4} \neq 2$ we conclude that they do not intersect.

Extension.

Method 2. (Determinants and the cross product). (Note these are no longer in the draft syllabus.)

It can be solved by a single calculation of a single number called the scalar triple product.

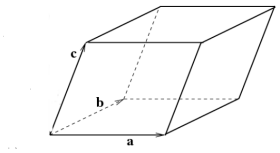
Firstly, a 3×3 determinant is $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)$

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ then the cross product

$\mathbf{b} \times \mathbf{c}$ is $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

The scalar triple product $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$ could be calculated by first finding the cross product $\mathbf{b} \times \mathbf{c}$ and then taking the dot product with $\mathbf{a}$. However there is a more efficient method because $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Geometrically the the absolute value of the scalar triple product represents the volume of the parallelepiped formed by the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$:



But a consequence of this is that if the vectors are coplanar then the scalar triple product will be 0.

Now pick 2 points each on ℓ_1 and ℓ_2, e.g., $A(0, 2, 1)$ and $B(2, 1, 3)$ on ℓ_1 and $C(1, 0, 3)$ and $D(3, 1, 4)$ on ℓ_2 (just by letting $\mu = 0, 1$ and $\lambda = 0, 1$ respectively) and find the scalar triple product $\overrightarrow{AB} \cdot \overrightarrow{AC} \times \overrightarrow{AD}$. If these three vectors are coplanar (which they will be if the lines intersect) then this scalar triple product will be 0.

Consider the following statement:

If the lines intersect then the scalar triple product will be 0.

The contrapositive of this statement is

If the scalar triple product is not 0 then the lines do not intersect.

So now solution 2 looks like this:

$\overrightarrow{AB} \cdot \overrightarrow{AC} \times \overrightarrow{AD} = \begin{vmatrix} 2 & -1 & 2 \\ 1 & -2 & 2 \\ 3 & -1 & 3 \end{vmatrix}$

$= 2(-2(3) - 2(-1)) - -1(1(3) - 2(3)) + 2(1(-1) - -2(3))$

$= -1$

$\neq 0$

Hence the lines do not intersect.

Minimum distance between skew lines

The lines are not parallel either because their direction vectors are not scalar multiples of each other. So we can also now ask what is the minimum distance between them. Notice that the cross product of two vectors is perpendicular to both. Indeed the interval between points, one on each line, will have minimum length precisely when it is perpendicular to both. Hence if those points are P, Q, then we have that
$P = (0 + 2\mu, 2 - \mu, 1 + 2\mu)$, $Q = (1 + 2\lambda, 0 + \lambda, 3 + \lambda)$ and so
$\overrightarrow{PQ} = \begin{pmatrix} 1+2\lambda-2\mu \\ \lambda+\mu-2 \\ \lambda-2\mu+2 \end{pmatrix} = k\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = k\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 2 \\ 2 & 1 & 1 \end{vmatrix}$
$= k\begin{pmatrix} -1(1)-2(1) \\ -(2(1)-2(2)) \\ 2(1)--1(2) \end{pmatrix} = \begin{pmatrix} -3k \\ 2k \\ 4k \end{pmatrix} \therefore \begin{pmatrix} 2\lambda-2\mu+3k \\ \lambda+\mu-2k \\ \lambda-2\mu-4k \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$

Solving this we get $\lambda = \frac{22}{29}$, $\mu = \frac{38}{29}$, $k = \frac{1}{29}$ and hence $\overrightarrow{PQ} = \frac{1}{29}\begin{pmatrix} -3 \\ 2 \\ 4 \end{pmatrix}$ which has length $\frac{\sqrt{29}}{29}$ and so this is the minimum distance between the skew lines.

It also raises the question if $\overrightarrow{AB} \cdot \overrightarrow{AC} \times \overrightarrow{AD} = 0$ does that necessarily mean the lines intersect?

Not necessarily because they could also be parallel (hence coplanar in which case we would still have $\overrightarrow{AB} \cdot \overrightarrow{AC} \times \overrightarrow{AD} = 0$.

So in this case we just have to check they are not parallel which is easy if you know the direction vectors are not scalar multiples of each other.

Also, if they did intersect, how to find the point of intersection?

Another extension (matrices) (Note again these have been removed from the draft syllabus)

Generally a line can be represented by 2 intersecting planes. So the 2 intersecting lines can be represented by 2 pairs of intersecting planes.
$a_{11}x + a_{12}y + a_{13}z = b_1$
$a_{21}x + a_{22}y + a_{23}z = b_2$
$a_{31}x + a_{32}y + a_{33}z = b_3$
$a_{41}x + a_{42}y + a_{43}z = b_4$
which can then be represented in a matrix equation $A\mathbf{w} = \mathbf{b}$ where $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}$, and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$. If the lines intersect then $\mathbf{w} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (A^TA)^{-1}A^T\mathbf{b}$ gives the point of intersection (x, y, z), where A^T is the transpose of A, i.e., the matrix resulting from swapping rows and columns.

Example 5.

Do $\ell_1 : \mathbf{x}_1 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ and $\ell_2 : \mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ intersect? If so, find the point of intersection.

Method 1.

If $\mathbf{x_1} = \mathbf{x}_2$ then $\mu\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} - \lambda\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

So $2\mu - 3\lambda = 1$ and $-\mu + \lambda = -1$. Solving simultaneously, $\mu = 2$ and $\lambda = 1$.

Now check $2\mu - 2\lambda = 2$ hence they intersect.

Substituting into either line with $\mu = 2$ or $\lambda = 1$, the point of intersection $(x, y, z) = (4, 0, 5)$.

Method 2. Choose 4 points, 2 on each line, e.g., with $\mu = 0, 1$ and $\lambda = 0, 1$, $A(0, 2, 1), B(2, 1, 3), C(1, 1, 3), D(4, 0, 5)$ then

$\overrightarrow{AB} \cdot \overrightarrow{AC} \times \overrightarrow{AD} = \begin{vmatrix} 2 & -1 & 2 \\ 1 & -1 & 2 \\ 4 & -2 & 4 \end{vmatrix}$

$= 2(-1(4) - 2(-2)) - -1(1(4) - 2(4)) + 2(1(-2) - -1(4))$

$= 0$

and the direction vectors $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ are not scalar multiples of each other. Hence they are non-parallel coplanar lines and therefore they intersect.

Rewriting the lines in Cartesian form
$\mu = \frac{x-0}{2} = \frac{y-2}{-1} = \frac{z-1}{2}$, $\lambda = \frac{x-1}{3} = \frac{y-1}{-1} = \frac{z-3}{2}$
and so 4 intersecting planes are
$x + 2y + 0z = 4$
$0x + 2y + z = 5$
$x + 3y + 0z = 4$
$0x + 2y + z = 5$
and so if $M = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 1 & 3 & 0 \\ 0 & 2 & 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 4 \\ 5 \\ 4 \\ 5 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then (x, y, z) is the point of intersection and $M\mathbf{w} = \mathbf{b}$.
Hence $\mathbf{w} = (M^T M)^{-1} M^T \mathbf{b}$

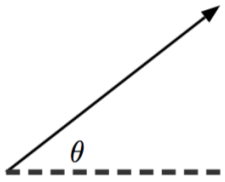
$$= \left(\begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 2 & 3 & 2 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 1 & 3 & 0 \\ 0 & 2 & 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 2 & 3 & 2 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix}$$

and hence $(x, y, z) = (4, 0, 5)$

Projectile Motion

Generally we have time-parametrised displacement vector $\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, velocity vector $\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$ and acceleration vector $\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \begin{pmatrix} x''(t) \\ y''(t) \end{pmatrix}$ and we also have Newton's second law that force is mass times acceleration, $\mathbf{F} = m\mathbf{a}$.

Now applying it to projectiles in 2 dimensions, we launch a projectile at an angle θ from the horizontal with initial speed $v_0 \text{ms}^{-1}$.



We now have $\mathbf{a} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$ where $g = 9.8\text{ms}^{-2}$ and initial velocity vector $\mathbf{v}_0 = \begin{pmatrix} v_0 \cos\theta \\ v_0 \sin\theta \end{pmatrix}$

If it launches from $(0, 0)$ then we get by integrating twice that $\mathbf{v} = \begin{pmatrix} v_0 \cos\theta \\ -gt + v_0 \sin\theta \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} v_0 \cos\theta t \\ -\frac{1}{2}gt^2 + v_0 \sin\theta t \end{pmatrix}$

So now we can ask questions about it like

When does the projectile reach the greatest height?

$y'(t) = -gt + v_0 \sin\theta = 0$ and hence $t = \frac{v_0 \sin\theta}{g}$ seconds.

What is the greatest height reached?

From the previous question t will be $\frac{v_0 \sin\theta}{g}$ and so

$y(t) = -\frac{1}{2}g(\frac{v_0 \sin\theta}{g})^2 + v_0 \sin\theta \frac{v_0 \sin\theta}{g} = \frac{v_0^2 \sin^2\theta}{2g}$

What angle gives greatest range?

Solving $y(t) = -\frac{1}{2}gt^2 + v_0 \sin\theta t = 0$ and $t \neq 0$ we get $t = \frac{2v_0 \sin\theta}{g}$

and hence $x(t) = \frac{2v_0^2 \sin\theta\cos\theta}{g} = \frac{v_0^2 \sin 2\theta}{g}$ which is a maximum when

$\sin 2\theta = 1$ and so $\theta = 45°$.

How fast is the projectile going when it hits the ground?

From the previous question when it hit the ground, $t = \frac{2v_0 \sin\theta}{g}$ and

so $\mathbf{v} = \begin{pmatrix} v_0\cos\theta \\ -v_0\sin\theta \end{pmatrix}$ which has magnitude

$|\mathbf{v}| = \sqrt{v_0^2\cos^2\theta + v_0^2\sin^2\theta} = v_0 \text{ms}^{-1}$,

i.e., the same speed it started with.

Use of Technology
Some scientific calculators have vector functionality and are already approved by NESA, e.g. the CASIO fx-100AU PLUS

Example 1.

Find $\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ using the CASIO fx-100AU PLUS.

Press MODE6121=2=SHIFT52223=4=AC SHIFT53+SHIFT54=

The result is

AnS
[4 6]

i.e., $\begin{pmatrix} 4 \\ 6 \end{pmatrix}$ as expected.

Example 2.

Find $3\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ using the CASIO fx-100 AU PLUS.

Given that we already entered $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ into memory, press

AC 3 × SHIFT 53=

Output is

AnS
[3 6]

i.e., $\begin{pmatrix} 3 \\ 6 \end{pmatrix}$ as expected.

Example 3. Find $3\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ on the CASIO fx-100 AU PLUS

We also previously entered $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and so we use

AC 3 × SHIFT 53-SHIFT 54=

Output:

AnS
[0 2]

i.e., $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ as expected.

Example 4.

Find $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ using the CASIO fx-100AU PLUS.

Use

AC SHIFT 53 SHIFT 57 SHIFT 54=

Output:

VctA·VctB

11

i.e., $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 11$ as expected.

Example 5.

Find the length of $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ using the CASIO fx-100AU PLUS.

Enter

AC SHIFT 51312=−1=2=AC Abs SHIFT 55)=

Output:

Abs(VctC)

3

i.e., $\left|\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}\right| = 3$ as expected.

Example 6.

Find the angle between $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ using the CASIO fx-100AU PLUS.

Use

AC SHIFT cos SHIFT 53 SHIFT 57 SHIFT 54÷(Abs SHIFT 53)Abs SHIFT 54)))=

Output:

$\cos^{-1}$(VctA·VctB÷(Abs(VctA)Abs(VctB)))

10.30484647

(provided the calculator is in degree mode)

i.e., $\cos^{-1} \frac{1\times3+2\times4}{\sqrt{1^2+2^2}\sqrt{3^2+4^2}} \approx 10.30484647^\circ$ as expected.

Extension.

Question 7. Find $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ on the CASIO fx-100AU PLUS

Use

AC SHIFT 53 × SHIFT 54=

Output:

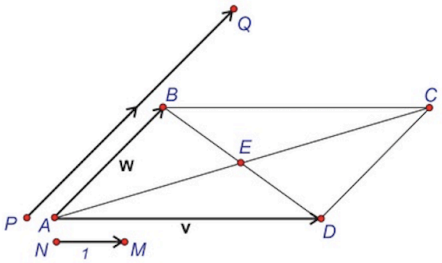
AnS
$[0 \quad 0 \quad -2]$

i.e., $\begin{pmatrix}1\\2\end{pmatrix} \times \begin{pmatrix}3\\4\end{pmatrix} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\1 & 2 & 0\\3 & 4 & 0\end{vmatrix} = \begin{pmatrix}0\\0\\-2\end{pmatrix}$ as expected.

Exam Questions.

Matching Question.

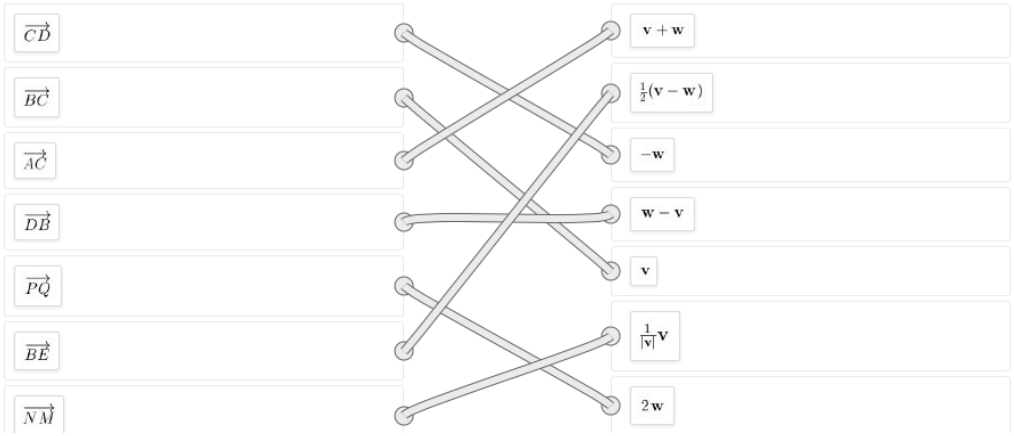
We are given the diagram:



Now match these up:

$\overrightarrow{CD}$	$\mathbf{v}+\mathbf{w}$
$\overrightarrow{BC}$	$\frac{1}{2}(\mathbf{v}-\mathbf{w})$
$\overrightarrow{AC}$	$-\mathbf{w}$
$\overrightarrow{DB}$	$\mathbf{w}-\mathbf{v}$
$\overrightarrow{PQ}$	$\mathbf{v}$
$\overrightarrow{BE}$	$\frac{1}{\lvert\mathbf{v}\rvert}\mathbf{v}$
$\overrightarrow{NM}$	$2\mathbf{w}$

Solution:



Multiple Choice.

A unit vector perpendicular to $5\mathbf{i}+\mathbf{j}-2\mathbf{k}$ is

A. $\frac{1}{4}(5\mathbf{i}+\mathbf{j}-2\mathbf{k})$

B. $2\mathbf{i}-4\mathbf{j}+3\mathbf{k}$

C. $\frac{1}{29}(2\mathbf{i}-4\mathbf{j}+3\mathbf{k})$

D. $\frac{1}{\sqrt{29}}(2\mathbf{i}-4\mathbf{j}+3\mathbf{k})$

E. $\frac{1}{\sqrt{30}}(5\mathbf{i}+\mathbf{j}-2\mathbf{k})$

Correct answer is D because the dot product with the given vector is 0 but the length is 1.

Free Response Question.
A particle moves so that its velocity at time t is given by

$$\mathbf{v}=-4\sin(2t)\mathbf{i}+6\cos(2t)\mathbf{j} \quad \text{for } 0\le t\le \tfrac{\pi}{2}.$$

a. Given that $\mathbf{r}(0)=2\mathbf{i}$, find the position vector $\mathbf{r}(t)$ of the particle at any time t.

b. Find the cartesian equation of the path followed by the particle.

c. Sketch the path followed by the particle.

a. $\mathbf{r}(t) = \int(-4\sin(2t)\mathbf{i} + 6\cos(2t)\mathbf{j})dt = 2\cos(2t)\mathbf{i} + 3\sin(2t)\mathbf{j} + \mathbf{C}$ and since $\mathbf{r}(0) = 2\mathbf{i}$, $\mathbf{C} = \mathbf{0}$ $\therefore \mathbf{r}(t) = 2\cos(2t)\mathbf{i} + 3\sin(2t)\mathbf{j}$.

b. $x = 2\cos(2t)$, $y = 3\sin(2t)$ and $\therefore \frac{x^2}{4} = \cos^2(2t)$ and $\frac{y^2}{9} = \sin^2(2t)$ $\therefore \frac{x^2}{4} + \frac{y^2}{9} = 1$.

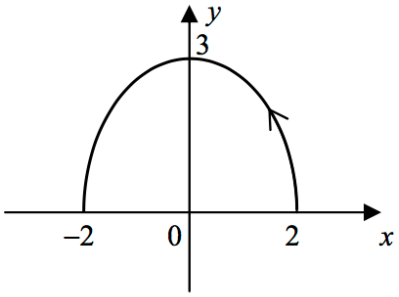
Given $0 \leq t \leq \frac{\pi}{2}$, then when $t = 0$, $x = 2$, $y = 0$

When $t = \frac{\pi}{4}$, $x = 0$, $y = 3$

When $t = \frac{\pi}{2}$, $x = -2$, $y = 0$

$\therefore -2 \leq x \leq 2$ and $0 \leq y \leq 3$

c.



More Exam Questions

Multiple Choice

1. The position vector of a particle at time $t \geq 0$ is given by $\mathbf{r} = (1 + t)\mathbf{i} + (1 - t)\mathbf{j}$. The path of the particle has equation

A. $y = x - 2$

B. $y = x + 2$

C. $y = -x - 2$

D. $y = -x + 2$

E. $y = x - 1$

2. Two particles, R and S, have position vectors $\mathbf{r} = (2t-10)\mathbf{i}+3\mathbf{j}$ and $\mathbf{s} = 2\mathbf{i}+(t-1)\mathbf{j}$ respectively at time t seconds, $t \geq 0$.

Then

A. R and S are in the same position when $t = 1$.

B. R and S are in the same position when $t = 4$.

C. R and S are in the same position when $t = 5$.

D. R and S are in the same position when $t = 6$.

E. R and S are never in the same position.

3. The position vector of a particle at time t seconds, $t \geq 0$, is given by $\mathbf{r}(t) = (3 - t)\mathbf{i} - 6\sqrt{t}\mathbf{j} + 5\mathbf{k}$. The direction of motion of the particle when $t = 9$ is

A. $-6\mathbf{i} - 18\mathbf{j} + 5\mathbf{k}$

B.$-\mathbf{i} - \mathbf{j}$

C. $-6\mathbf{i} - \mathbf{j}$

D. $-\mathbf{i} - \mathbf{j} + 5\mathbf{k}$

E. $-13.5\mathbf{i} - 108\mathbf{j} + 45\mathbf{k}$

4. Let $\mathbf{u} = \mathbf{i} + \mathbf{j}$ and $\mathbf{v} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$. The angle between the vectors $\mathbf{u}$ and $\mathbf{v}$ is

A. $0°$

B. $45°$

C. $30°$

D. $22.5°$

E. $90°$

5. In the cartesian plane, a vector perpendicular to the line $3x + 2y + 1 = 0$ is

A. $3\mathbf{i} + 2\mathbf{j}$

B. $-\frac{1}{2}\mathbf{i} + \frac{1}{3}\mathbf{j}$

C. $2\mathbf{i} - 3\mathbf{j}$

D. $\frac{1}{2}\mathbf{i} - \frac{1}{3}\mathbf{j}$

E. $2\mathbf{i} + 3\mathbf{j}$

6. A force of magnitude 18 newtons acts on a body at an angle of 150° in the anti-clockwise direction to the vector $\mathbf{i}$. A vector representation of this force could be

A. $9\sqrt{3}\mathbf{i} + 9\mathbf{j}$

B. $-9\mathbf{i} + 9\sqrt{3}\mathbf{j}$

C. $-9\sqrt{3}\mathbf{i} + 9\mathbf{j}$

D. $-9\mathbf{i} - 9\sqrt{3}\mathbf{j}$

E. $9\sqrt{3}\mathbf{i} - 9\mathbf{j}$

7. The angle between the vectors $\mathbf{a} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ is best represented by

A. $-\frac{4}{9}$

B. $-\cos^{-1}(\frac{4}{9})$

C. $\pi + \cos^{-1}(-\frac{4}{9})$

D. $\pi - \cos^{-1}(\frac{4}{9})$

E. $\cos^{-1}(\pi - \frac{4}{9})$

8. Let $\mathbf{u} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = a\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. If the scalar resolute of $\mathbf{v}$ in the direction of $\mathbf{u}$ is 1, then the value of a is

A. $-\frac{3}{2}$

B. $-\frac{2}{3}$

C. 3

D. $\frac{2}{3}$

E. $\frac{3}{2}$

9. A cricket ball is hit from an origin a ground level so that its position vector at time t is given by $\mathbf{r}(t) = 15t\mathbf{i} + (20t - 5t^2)\mathbf{j}$ for $t \geq 0$, where $\mathbf{i}$ is a unit vector in the forward direction and $\mathbf{j}$ is a unit vector vertically up. When the cricket ball reaches its maximum height, its position vector is

A. $\mathbf{r} = 20\mathbf{i} + 30\mathbf{j}$

B. $\mathbf{r} = 15\mathbf{i} + 20\mathbf{j}$

C. $\mathbf{r} = 60\mathbf{i}$

D. $\mathbf{r} = 30\mathbf{i} + 10\mathbf{j}$

E. $\mathbf{r} = 30\mathbf{i} + 20\mathbf{j}$

10. If the vectors $\mathbf{a} = m\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = m\mathbf{i} + m\mathbf{j} - 4\mathbf{k}$ are perpendicular, then

A. $m = 0$

B. $m = -6$ or $m = 2$

C. $m = -2$ or $m = 6$

D. $m = -2$ or $m = 0$

E. $m = -1$ or $m = 1$

11. Two forces $\mathbf{P}$ and $\mathbf{Q}$ act on a body. $\mathbf{P}$ acts in the direction of $\mathbf{i}$ with magnitude one newton and $\mathbf{Q}$ acts in the direction of $\mathbf{i} + \sqrt{3}\mathbf{j}$ with magnitude of four newtons. The magnitude of the total force acting on the body, in newtons, is

A. 3

B. $\sqrt{15}$

C. $\sqrt{17}$

D. $\sqrt{21}$

E. 5

12. P, Q and R are three collinear points with position vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ respectively, where Q lies between P and R. If $|\overrightarrow{QR}| = \frac{1}{2}|\overrightarrow{PQ}|$, then $\mathbf{r}$ is equal to

A. $\frac{3}{2}\mathbf{q} - \frac{1}{2}\mathbf{p}$

B. $\frac{3}{2}\mathbf{p} - \frac{1}{2}\mathbf{q}$

C. $\frac{3}{2}\mathbf{q} - \frac{3}{2}\mathbf{p}$

D. $\frac{1}{2}\mathbf{p} - \frac{3}{2}\mathbf{q}$

E. $\frac{3}{2}\mathbf{p} - \frac{3}{2}\mathbf{q}$

13. A force $\mathbf{F}$ is applied to a body causing it to accelerate in the direction of vector $\mathbf{d}$. The magnitude of the force which causes the body to accelerate in this direction is given by

A. $\mathbf{F} \cdot \mathbf{d}$

B. $\frac{\mathbf{d}}{\mathbf{F}}$

C. $\frac{\mathbf{F} \cdot \mathbf{d}}{|\mathbf{F}|}$

D. $\frac{\mathbf{F}}{\mathbf{d}}$

E. $\frac{\mathbf{F} \cdot \mathbf{d}}{|\mathbf{d}|}$

14. Consider the three vectors $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = -3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ and $\mathbf{c} = 13\mathbf{i} + 10\mathbf{j} + \mathbf{k}$. It follows that

A. $\mathbf{c}$ and $\mathbf{b}$ are perpendicular to $\mathbf{a}$

B. $\mathbf{c}$ is only perpendicular to $\mathbf{b}$

C. $\mathbf{c}$ is only perpendicular to $\mathbf{a}$

D. $\mathbf{a}$ and $\mathbf{b}$ are perpendicular to $\mathbf{c}$

E. $\mathbf{a}$ is only perpendicular to $\mathbf{b}$

15. The position vector of a particle at time $t \geq 0$ is given by $\mathbf{r} = \sin(t)\mathbf{i} + \cos(2t)\mathbf{j}$. The path of the particle has cartesian equation

A. $y = 2x^2 - 1$

B. $y = 1 - 2x^2$

C. $y = \sqrt{1 - x^2}$

D. $y = \sqrt{x^2 - 1}$

E. $y = 2x\sqrt{1 - x^2}$

16. The scalar resolute of $\mathbf{a} = 3\mathbf{i} - \mathbf{k}$ in the direction of $\mathbf{b} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ is

A. $\frac{8}{\sqrt{10}}$

B. $\frac{8}{9}(2\mathbf{i} - \mathbf{j} - 2\mathbf{k})$

C. 8

D. $\frac{4}{5}(3\mathbf{i} - \mathbf{k})$

E. $\frac{8}{3}$

17. The square of the magnitude of the vector $\mathbf{d} = 5\mathbf{i} - \mathbf{j} + \sqrt{10}\mathbf{k}$ is

A. 6

B. 34

C. 36

D. 51.3

E. $\sqrt{34}$

18. The angle between the vectors $\mathbf{a} = \mathbf{i} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + \mathbf{j}$ is exactly

A. $\frac{\pi}{6}$

B. $\frac{\pi}{4}$

C. $\frac{\pi}{3}$

D. $\frac{\pi}{2}$

E. π

19. Consider the three forces $\mathbf{F}_1 = -\sqrt{3}\mathbf{j}$, $\mathbf{F}_2 = \mathbf{i} + \sqrt{3}\mathbf{j}$ and $\mathbf{F}_3 = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$. The magnitude of the sum of these three forces is equal to

A. the magnitude of $(\mathbf{F}_3 - \mathbf{F}_1)$

B. the magnitude of $(\mathbf{F}_2 - \mathbf{F}_1)$

C. the magnitude of $\mathbf{F}_1$

D. the magnitude of $\mathbf{F}_2$

E. the magnitude of $\mathbf{F}_3$

20. The angle between the vectors $3\mathbf{i} + 6\mathbf{j} - \mathbf{k}$ and $2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, correct to the nearest tenth of a degree, is

A. 2.0°

A. 2.0°

B. 91.0°

C. 112.4°

D. 121.3°

E. 124.9°

21. The position of a particle at time t is given by $\mathbf{r}(t) = (\sqrt{t-2})\mathbf{i} + (2t)\mathbf{j}$ for $t \geq 2$. The cartesian equation of the path of the particle is

A. $y = 2x^2 + 4$, $x \geq 2$

B. $y = 2x^2 + 2$, $x \geq 2$

C. $y = 2x^2 + 4,\ x \geq 0$

D. $y = \sqrt{\frac{x-4}{2}},\ x \geq 4$

E. $y = 2x^2 + 2,\ x \geq 0$

22. The vectors $\mathbf{a} = 2\mathbf{i} + m\mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = m^2\mathbf{i} - \mathbf{j} + \mathbf{k}$ are perpendicular for

A. $m = -\frac{2}{3}$ and $m = 1$

B. $m = -\frac{3}{2}$ and $m = 1$

C. $m = \frac{2}{3}$ and $m = -1$

D. $m = \frac{3}{2}$ and $m = -1$

E. $m = 3$ and $m = -1$

23. If $\mathbf{u} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$, the vector resolute of $\mathbf{v}$ in the direction of $\mathbf{u}$ is

A. $\frac{20}{49}(3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})$

B. $\frac{20}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

C. $\frac{20}{7}(3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})$

D. $\frac{20}{9}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

E. $\frac{1}{9}(-2\mathbf{i} + 2\mathbf{j} - \mathbf{k})$

24. If θ is the angle between $\mathbf{a} = \sqrt{3}\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 4\mathbf{j} + \sqrt{3}\mathbf{k}$, then $\cos(2\theta)$ is

A. $-\frac{4}{5}$

B. $\frac{7}{25}$

C. $-\frac{7}{25}$

D. $\frac{14}{25}$

E. $-\frac{24}{25}$

25. Two vectors are given by $\mathbf{a} = 4\mathbf{i} + m\mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = -2\mathbf{i} + n\mathbf{j} - \mathbf{k}$, where $m, n \in \mathbb{R}^+$. If $|\mathbf{a}| = 10$ and $\mathbf{a}$ is perpendicular to $\mathbf{b}$, then m and n respectively are

A. $5\sqrt{3}, \frac{\sqrt{3}}{3}$

B. $5\sqrt{3}, \sqrt{3}$

C. $-5\sqrt{3}, \sqrt{3}$

D. $\sqrt{93}, \frac{5\sqrt{93}}{93}$

E. 5, 1

26. The acceleration vector of a particle that stars from rest is given by $\mathbf{a}(t) = -4\sin(2t)\mathbf{i} + 20\cos(2t)\mathbf{j} - 20e^{-2t}\mathbf{k}$, where $t \geq 0$. The velocity vector of the particle, $\mathbf{v}(t)$ is given by

A. $-8\cos(2t)\mathbf{i} - 40\sin(2t)\mathbf{j} + 40e^{-2t}\mathbf{k}$

B. $2\cos(2t)\mathbf{i} + 10\sin(2t)\mathbf{j} + 10e^{-2t}\mathbf{k}$

C. $(8 - 8\cos(2t))\mathbf{i} - 40\sin(2t)\mathbf{j} + (40e^{-2t} - 40)\mathbf{k}$

D. $(2\cos(2t) - 2)\mathbf{i} + 10\sin(2t)\mathbf{j} + (10e^{-2t} - 10)\mathbf{k}$

E. $(4\cos(2t) - 4)\mathbf{i} + 20\sin(2t)\mathbf{j} + (20 - 20e^{-2t})\mathbf{k}$

27. The velocity vector of a 5 kg mass moving in the cartesian plane is given by $\mathbf{v}(t) = 3\sin(2t)\mathbf{i} + 4\cos(2t)\mathbf{j}$, where velocity components are measured in m/s. During its motion, the maximum magnitude of the net force, in newtons, acting on the mass is

A. 8

B. 30

C. 40

D. 50

E. 70.

28. The component of the force $\mathbf{F} = a\mathbf{i} + b\mathbf{j}$, where a and b are non-zero real constants, in the direction of the vector $\mathbf{w} = \mathbf{i} + \mathbf{j}$, is

A. $\left(\frac{a+b}{2}\right)\mathbf{w}$

B. $\frac{\mathbf{F}}{a+b}$

C. $\left(\frac{a+b}{a^2+b^2}\right)\mathbf{F}$

D. $(a+b)\mathbf{w}$

E. $\left(\frac{a+b}{\sqrt{2}}\right)\mathbf{w}$

29. Points A, B and C have position vectors $\mathbf{a} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{b} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{c} = -3\mathbf{j} + \mathbf{k}$ respectively. The cosine of angle ABC is equal to

A. $\frac{5}{\sqrt{6}\sqrt{10}}$

B. $\frac{7}{\sqrt{6}\sqrt{13}}$

C. $-\frac{1}{\sqrt{6}\sqrt{13}}$

D. $-\frac{7}{\sqrt{21}\sqrt{6}}$

E. $-\frac{2}{\sqrt{6}\sqrt{13}}$

30. The position vectors of two moving particles are given by $\mathbf{r}_1(t) = (2+4t^2)\mathbf{i} + (3t+2)\mathbf{j}$ and $\mathbf{r}_2(t) = (6t)\mathbf{i} + (4+t)\mathbf{j}$, where $t \geq 0$. The particles will collide at

A. $3\mathbf{i} + 3.5\mathbf{j}$

B. $6\mathbf{i} + 5\mathbf{j}$

C. $3\mathbf{i} + 4.5\mathbf{j}$

D. $0.5\mathbf{i} + \mathbf{j}$

E. $5\mathbf{i} + 6\mathbf{j}$

31. Let $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + \alpha\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} - \mathbf{j} + \alpha^2\mathbf{k}$, where α is a real constant. If the scalar resolute of $\mathbf{a}$ in the direction of $\mathbf{b}$ is $\frac{74}{\sqrt{273}}$, then α equals

A. 1

B. 2

C. 3

D. 4

E. 5

32. If $\mathbf{a} = -2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = -m\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, where m is a real constant, the vector $\mathbf{a} - \mathbf{b}$ will be perpendicular to vector $\mathbf{b}$ where m equals

A. 0 only

B. 2 only

C. 0 or 2

D. 4.5

E. 0 or -2

33. A particle of mass 5 kg is subject to forces $12\mathbf{i}$ newtons and $9\mathbf{j}$ newtons. If no other forces act on the particle, the magnitude of the particle's acceleration, in ms^{-2}, is

A. 3

B. $2.4\mathbf{i} + 1.8\mathbf{j}$

C. 4.2

D. 9

E. $60\mathbf{i} + 45\mathbf{j}$

Answers

1. D **2.** E **3.** B **4.** B **5.** A **6.** C **7.** D **8.** E **9.** E **10.** B **11.** D
12. A **13.** E **14.** D **15** B **16.** E **17.** C **18.** C **19.** E **20.** C **21.** C **22.** D
23. D **24.** B **25.** A **26.** D **27.** C **28.** A **29.** C **30.** B **31.** D **32.** C **33.** A

Free Response

1. The position vector of a moving particle is given by $\mathbf{r}(t) = \sqrt{t-2}\mathbf{i} + 2t\mathbf{j}$ for $2 \leq t \leq 6$.

a. Find the cartesian equation of the path followed by the particle.

b. Sketch the path of the particle.

2. Point A has position vector $\mathbf{a} = -\mathbf{i} - 4\mathbf{j}$, point B has position vector $\mathbf{b} = 2\mathbf{i} - 5\mathbf{j}$, point C has position vector $\mathbf{c} = 5\mathbf{i} - 4\mathbf{j}$, and point D has position vector $\mathbf{d} = 2\mathbf{i} + 5\mathbf{j}$ relative to the origin O.

a. Show that $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are perpendicular,

b. Use a vector method to find the cosine of $\angle ADC$, the angle between $\overrightarrow{DA}$ and $\overrightarrow{DC}$.

c. Find the cosine of $\angle ABC$, and hence show that $\angle ADC$ and $\angle ABC$ are supplementary.

d. Point P has position vector $\mathbf{p} = 2\mathbf{i}$. Use the cosine of $\angle APC$ and an appropriate trigonometric formula to prove that $\angle APC = 2\angle ADC$.

3. A particle moves in the plane with position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ where x and y are functions of t. If its velocity vector is $\mathbf{v} = -y\mathbf{i} + x\mathbf{j}$, find the acceleration vector of the particle in terms of the position vector $\mathbf{r}$.

4. An aircraft approaching an airport with velocity $\mathbf{v} = 30\mathbf{i} - 40\mathbf{j} - 4\mathbf{k}$ is observed on the control tower radar screen at time $t = 0$ seconds. Ten seconds later it passes over a navigation beacon with position vector $-500\mathbf{i} + 2500\mathbf{j}$ relative to the base of the control tower, at an altitude of 200 metres. Let $\mathbf{i}$ and $\mathbf{j}$ be horizontal orthogonal unit vectors and let $\mathbf{k}$ be a unit vector in the vertical direction. Displacement components are measured in metres.

a. Show that the position vector of the aircraft relative to the base of the control tower at time t is given by $\mathbf{r}(t) = (30t - 800)\mathbf{i} + (2900 - 40t)\mathbf{j} + (240 - 4t)\mathbf{k}$

b. When does the aircraft land and how far (correct to the nearest metre) from the base of the control tower is the point of landing?

c. At what angle from the runway, correct to the nearest tenth of a degree, does the aircraft land?

d. At what time, correct to the nearest second, is the aircraft closest to the base of the control tower?

e. What distance does the aircraft travel from the time it is observed on the radar screen to the time it lands? Give your answer correct to the nearest metre.

5. The coordinates of three points are $A(1, 0, 5), B(-1, 2, 4)$ and $C(3, 5, 2)$.

a. Express the vector $\overrightarrow{AB}$ in the form $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

b. Find the coordinates of the point D such that $ABCD$ is a parallelogram.

c. Prove that $ABCD$ is a rectangle.

6. Relative to an origin O, point A has cartesian coordinates $(1, 2, 2)$ and point B has cartesian coordinates $(-1, 3, 4)$.

a. Find an expression for the vector $\overrightarrow{AB}$ in the form $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

b. Show that the cosine of the angle between the vectors **OA** and **AB** is $\frac{4}{9}$

c. Hence find the exact area of the triangle OAB.

7. The path of a particle is given by $\mathbf{r}(t) = t\sin(t)\mathbf{i} - t\cos(t)\mathbf{j},\ t \geq 0$. The particle leaves the origin at $t = 0$ and then spirals outwards.

a. Show that the second time the particle crosses the x-axis after leaving the origin occurs when $t = \frac{3\pi}{2}$.

b. Find the speed of the particle when $t = \frac{3\pi}{2}$.

c. Let θ be the acute angle at which the path of the particle crosses the x-axis. Find $\tan(\theta)$ when $t = \frac{3\pi}{2}$.

8. A golfer hits a ball at time $t = 0$ seconds from an origin O, aiming at a hole which is 200 metres away at the end of a horizontal fairway. The initial velocity of the ball is

given by $\mathbf{v}_0 = 35\mathbf{i} + 5\mathbf{j} + 24.5\mathbf{k}$, where $\mathbf{i}$ is a unit vector in the direction of the hole, $\mathbf{j}$ is a horizontal unit vector to the left perpendicular to $\mathbf{i}$, and $\mathbf{k}$ is a unit vector vertically up. Velocity components are measured in metres per second. The ball, once in the air, is subject only to gravitational acceleration.

a. Given that the acceleration of the ball is $\mathbf{a}(t) = -9.8\mathbf{k}$ show by integration that the position vector of the ball t seconds after the golfer hits it is $\mathbf{r}(t) = 35t\mathbf{i} + 5t\mathbf{j} + (24.5t - 4.9t^2)\mathbf{k}$.

b. Show that the ball is in the air for five seconds.

c. Find the maximum height, in metres, reached by the ball.

d. Find the speed of the ball when it hits the ground. Give your answer in metres per second, correct to the nearest integer.

e. Find the distance from the hole to where the ball hits the ground. Give your answer correct to the nearest metre.

9. The position of a particle at time t is given by $\mathbf{r}(t) = (2\sqrt{t^2+2} - t^2)\mathbf{i} + (2\sqrt{t^2+2} + 2t)\mathbf{j}$, $t \geq 0$.

a. Find the velocity of the particle at time t.

b. Find the speed of the particle at time $t = 1$ in the form $\frac{a\sqrt{b}}{c}$, where a, b and c are positive integers.

c. Show that at time $t = 1$, $\frac{dy}{dx} = \frac{1+\sqrt{3}}{1-\sqrt{3}}$.

d. Find the angle in terms of π, between the vector $-\sqrt{3}\mathbf{i} + \mathbf{j}$ and the vector $\mathbf{r}(t)$ at time $t = 0$.

10. The coordinates of three points are $A(-1, 2, 4)$, $B(1, 0, 5)$ and $C(3, 5, 2)$.

a. Find $\overrightarrow{AB}$

b. The points A, B and C are the vertices of a triangle. Prove that the triangle has a right angle at A.

c. Find the length of the hypotenuse of the triangle.

11. The position vector $\mathbf{r}(t)$ of a particle moving relative to an origin O at time t seconds is given by $\mathbf{r}(t) = 4\sec(t)\mathbf{i} + 2\tan(t)\mathbf{j}$, $t \in [0, \frac{\pi}{2})$ where the components are measured in metres.

a. Show that the cartesian equation of the path of the particle is $\frac{x^2}{16} - \frac{y^2}{4} = 1$.

b. Sketch the path of the particle, labelling any asymptotes with their equations.

c. Find the speed of the particle, in ms^{-1}, when $t = \frac{\pi}{4}$.

12. Let $\mathbf{a} = -\frac{7\sqrt{3}}{3}\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + \sqrt{3}\mathbf{j} + 2\sqrt{3}\mathbf{k}$

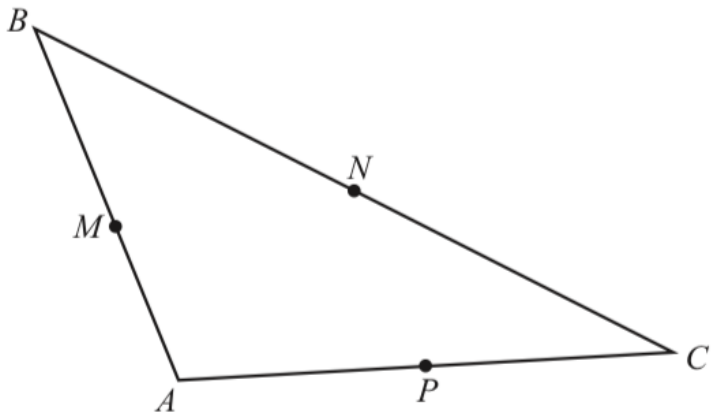
a. Find a unit vector in the direction of $\mathbf{b}$.

b. Resolve $\mathbf{a}$ into two vector components, one that is parallel to $\mathbf{b}$ and one that is perpendicular to $\mathbf{b}$.

c. Find the value of m such that $\mathbf{c} = m\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ makes an angle of $\frac{2\pi}{3}$ with $\mathbf{b}$ and where $\mathbf{c} \neq \mathbf{a}$.

d. Find the angle, in degrees, that $\mathbf{c}$ makes with $\mathbf{a}$, correct to one decimal place.

e. For the triangle ABC shown below, the midpoints of the sides are the points M, N and P.



i. Express $\overrightarrow{AN}$ in terms of $\mathbf{u}$ and $\mathbf{v}$.

ii. Express $\overrightarrow{CM}$ and $\overrightarrow{BP}$ in terms of $\mathbf{u}$ and $\mathbf{v}$.

iii. Hence simplify the expression $\overrightarrow{AN} + \overrightarrow{CM} + \overrightarrow{BP}$.

13. Consider the vector $\mathbf{a} = \sqrt{3}\mathbf{i} - \mathbf{j} - \sqrt{2}\mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors in the positive directions of the x, y and z axes respectively.

a. Find the unit vector in the direction of $\mathbf{a}$.

b. Find the acute angle that $\mathbf{a}$ makes with the positive direction of the x-axis.

c. The vector $\mathbf{b} = 2\sqrt{3}\mathbf{i} + m\mathbf{j} - 5\mathbf{k}$. Given that $\mathbf{b}$ is perpendicular to $\mathbf{a}$, find the value of m.

14. The position vector of a particle at time $t \geq 0$ is given by $\mathbf{r}(t) = (t-2)\mathbf{i} + (t^2 - 4t + 1)\mathbf{j}$.

a. Show that the cartesian equation of the path followed by the particle is $y = x^2 - 3$.

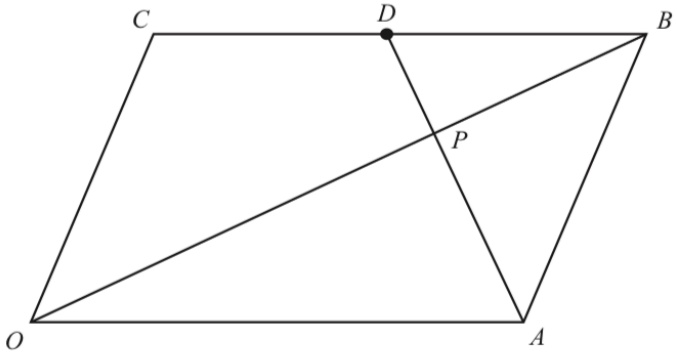
b. Sketch the path followed by the particle, labelling all important features.

c. Find the speed of the particle when $t = 1$.

15. Let $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$.

a. Express $\mathbf{a}$ as the sum of two vector resolutes, one of which is parallel to $\mathbf{b}$ and the other of which is perpendicular to $\mathbf{b}$. Identify clearly the parallel vector resolute and the perpendicular vector resolute.

b. $OABC$ ia a parallelogram where D is the midpoint of $\overline{CB}$. $\overline{OB}$ and $\overline{AD}$ intersect at point P. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$.

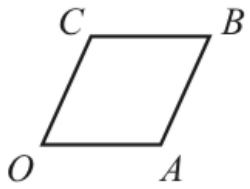


i. Given that $\overrightarrow{AP} = \alpha\overrightarrow{AD}$, write an expression for $\overrightarrow{AP}$ in terms of α, $\mathbf{a}$ and $\mathbf{c}$.

ii. Given that $\overrightarrow{OP} = \beta\overrightarrow{OB}$, write another expression for $\overrightarrow{AP}$ in terms of β, βa and $\mathbf{c}$.

iii. Hence deduce the values of α and β.

16. Consider the rhombus $OABC$ shown below, where $\overrightarrow{OA} = a\mathbf{i}$ and $\overrightarrow{OC} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, and a is a positive real constant.



a. Find a.

b. Show that the diagonals of the rhombus $OABC$ are perpendicular.

17. The velocity of a particle at time t seconds is given by $\dot{\mathbf{r}}(t) = (4t - 3)\mathbf{i} + 2t\mathbf{j} - 5\mathbf{k}$, where components are measured in metres per second. Find the distance of the particle from the origin in metres when $t = 2$, given that $\mathbf{r}(0) = \mathbf{i} - 2\mathbf{k}$.

18. The position vector $\mathbf{r}(t)$, from origin O, of a model helicopter t seconds after leaving the ground is given by $\mathbf{r}(t) = (50 + 25\cos(\frac{\pi t}{30}))\mathbf{i} + (50 + 25\sin(\frac{\pi t}{30}))\mathbf{j} + \frac{2t}{5}\mathbf{k}$ where $\mathbf{i}$ is a unit vector to the east, $\mathbf{j}$ is a unit vector to the north and $\mathbf{k}$ is a unit vector vertically up. Displacement components are measured in metres.

a i Find the time, in seconds, required for the helicopter to gain an altitude of 60 m.

ii Find the angle of elevation from O of the helicopter when it is at an altitude of 60 m. Give your answer in degrees, correct to the nearest degree.

b After how many seconds will the helicopter first be directly above the point of take-off?

c Show that the velocity of the helicopter is perpendicular to its acceleration.

d Find the speed of the helicopter in ms^{-1}, giving your answer correct to two decimal places.

e. A treetop has position vector $\mathbf{r} = 60\mathbf{i} + 40\mathbf{j} + 8\mathbf{k}$. Find the distance of the helicopter from the treetop after it has been travelling for 45 seconds. Give your answer in metres, correct to one decimal place.

19. The position of a body with mass 3 kg from a fixed origin at time t seconds, $t \geq 0$, is given by $\mathbf{r} = (3\sin(2t) - 2)\mathbf{i} + (3 - 2\cos(2t))\mathbf{j}$, where components are in metres.

a. Find an expression for the speed, in metres per second, of the body at time t.

b. Find the speed of the body, in metres per second, when $t = \frac{\pi}{12}$.

c. Find the maximum magnitude of the net force acting on the body in newtons.

20. Two ships, A and B, are observed from a lighthouse at origin O. Relative to O, their position vectors at time t hours after midday are given by $\mathbf{r}_A = 5(1-t)\mathbf{i}+3(1+t)\mathbf{j}$, $\mathbf{r}_B = 4(t-2)\mathbf{i} + (5t-2)\mathbf{j}$ where displacements are measured in kilometres.

a. Show that the two ships will not collide, clearly stating your reason.

b. Sketch and label the path of each ship. Show the direction of motion of each ship with an arrow.

c. Find the obtuse angle between the paths of the two ships. Give your answer in degrees, correct to one decimal place.

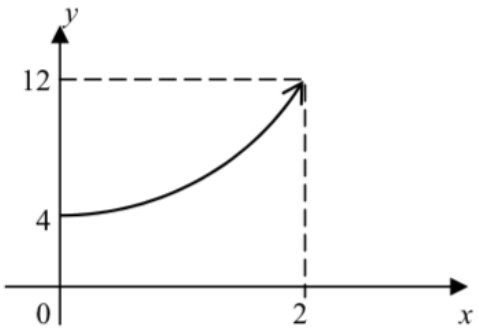
d. i. Find the value of t, correct to three decimal places, when the ships are closest.

ii. Find the minimum distance between the two ships, in kilometres, correct to two decimal places.

Answers.

1. a. $y = 2(x^2 + 2)$ for $0 \leq x \leq 2$ and $4 \leq y \leq 12$

b.



2. b. 0.8 **c.** -0.8

3. $-\mathbf{r}$

4. **b.** $t = 60$s, distance$= 1118$m. **c.** $4.6°$ **d.** 56s **e.** 3010m.

5. **a.** $-2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ **b.** $(5, 3, 3)$

6. **a.** $-2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ **c.** $\frac{\sqrt{65}}{2}$

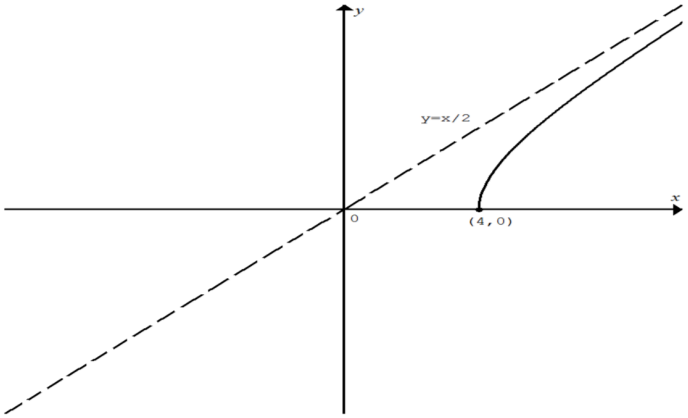
7. **b.** $\frac{\sqrt{4+9\pi^2}}{2}$ **c.** $\frac{3\pi}{2}$

8. **c.** 30.625m **d.** 43ms^{-1} **e.** 35m.

9. **a.** $(\frac{2t}{\sqrt{t^2+2}-2t})\mathbf{i} + (\frac{2t}{\sqrt{t^2+2}} + 2)\mathbf{j}$ **b.** $\frac{4\sqrt{6}}{3}$ **d.** $\frac{1+\sqrt{3}}{1-\sqrt{3}}$

10. **a.** $2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ **c.** $\sqrt{38}$

11. **b.**

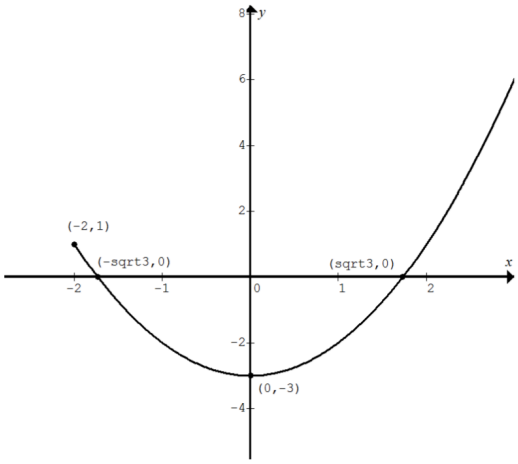


c. $4\sqrt{3}\text{ms}^{-1}$

12. **a.** $\frac{1}{4}(\mathbf{i} + \sqrt{3}\mathbf{j} + 2\sqrt{3})\mathbf{k}$ **b.** $-(\frac{\sqrt{3}}{3}\mathbf{i} + \mathbf{j} + 2\mathbf{k})$, $-2\sqrt{3}\mathbf{i} + 2\mathbf{j}$ **c.** $\frac{\sqrt{3}}{3}$ **d.** $75.5°$ **e.** **i.** $\mathbf{u} + \frac{1}{2}\mathbf{v}$ **ii.** $-\frac{1}{2}\mathbf{u} - \mathbf{v}$ **iii.** $\mathbf{0}$.

13. **a.** $\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}$ **b.** $\frac{\pi}{4}$ **c.** $6 + 5\sqrt{2}$

14. b.



c. $\sqrt{5}$

15. a. Parallel vector resolute $= \frac{1}{9}(2\mathbf{i} - 2\mathbf{j} - \mathbf{k})$, perpendicular vector resolute $= \frac{5}{9}(5\mathbf{i} + 4\mathbf{j} + 2\mathbf{k})$ and $\mathbf{a} = \frac{1}{9}(2\mathbf{i} - 2\mathbf{j} - \mathbf{k}) + \frac{5}{9}(5\mathbf{i} + 4\mathbf{j} + 2\mathbf{k})$
b. i. $\alpha\mathbf{c} - \frac{1}{2}\alpha\mathbf{a}$ **ii.** $\beta\mathbf{c} - (1-\beta)\mathbf{a}$ **iii.** $\frac{2}{3}$, $\frac{2}{3}$.

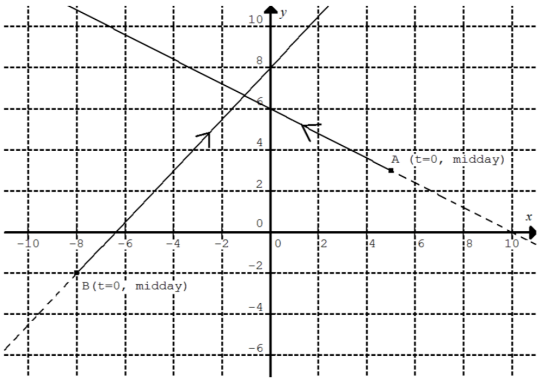
16. a. $\sqrt{3}$

17. a. 1

18. a. i. 150s **ii.** 47° **b.** 60s **d.** 2.65ms^{-1} **e.** 20.6m

19. a. $4\sqrt{\frac{5}{4}\cos^2 2t + 1}$m/s **b.** $\sqrt{31}$m/s **c.** 36N

20. b.



c. 97.7° **d. i.** 1.494 **ii.** 2.06km